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Calculus of Variations

Nonexistence of Ginzburg–Landau minimizers with prescribed degree on the boundary of a doubly connected domain

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Abstract

Let ω, Ω be bounded simply connected domains in $\mathbb{R}^2$, and let $\overline{\omega} \subset \Omega$. In the annular domain $A = \Omega \setminus \overline{\omega}$ we consider the class $\mathcal{J}$ of complex valued maps having modulus 1 and degree 1 on $\partial\Omega$ and $\partial\omega$.

We prove that, when $\mathrm{cap}(A) < \pi$, there exists a finite threshold value κ_1 of the Ginzburg–Landau parameter κ such that the minimum of the Ginzburg–Landau energy E_κ not attained in $\mathcal{J}$ when $\kappa > \kappa_1$ while it is attained when $\kappa < \kappa_1$. ***To cite this article: L. Berlyand et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Nonexistence des minimizers de Ginzburg–Landau avec le degré prescrit sur la frontière d'un domaine doublement connexe. Soient ω, Ω des ouverts bornés, simplement connexes de $\mathbb{R}^2$, et soit $\overline{\omega} \subset \Omega$. Dans le domaine annulaire $A = \Omega \setminus \overline{\omega}$ on considère une classe $\mathcal{J}$ des applications à valeurs complexes ayant le module égal à 1 et le degré 1 sur $\partial\Omega$ et $\partial\omega$.

On montre que, si $\mathrm{cap}(A) < \pi$, alors il existe une valeur critique finie κ_1 du paramètre κ de Ginzburg–Landau, telle que le minimum de l'énergie de Ginzburg–Landau E_κ n'est pas atteint dans $\mathcal{J}$ pour $\kappa > \kappa_1$, tandis qu'il est attaint pour $\kappa < \kappa_1$. ***Pour citer cet article : L. Berlyand et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

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On montre que, si $\mathrm{cap}(A) < \pi$, alors il existe une valeur critique finie κ_1 du paramètre κ de Ginzburg–Landau, telle que le minimum m_κ de l'énergie de Ginzburg–Landau E_κ n'est pas atteint dans $\mathcal{J}$ pour $\kappa > \kappa_1$, tandis qu'il est atteint pour $\kappa < \kappa_1$.

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La démonstration du résultat essentiel (voir Théorème 1.1) est basée sur l'éstimation $m_\kappa \leqslant 2\pi$ obtenue dans [7] ainsi que sur des résultats de convergence connus de [6].

On procède par contradiction. Supposons que le Théorème 1.1 est faux. Alors le minimiseur de E_κ existe pour tout κ finis.

On suppose d'abord que $\mathcal{A}$ est un anneau circulaire conforme équivalent à A. Sous l'hypothèse ci-déssus on montre que le minimum de l'énergie de Ginzburg–Landau pour le domaine $\mathcal{A}$ est atteint dans la classe $\mathcal{J}$ pour chaque κ. On peut supposer alors que A est un anneau circulaire.

Puisque m_κ est atteint pour tous κ il existe $u_\kappa \in \mathcal{J}$ tel que $E_\kappa[u_\kappa] = m_\kappa \leqslant 2\pi$.

Ensuite on construit une famille de fonctionnelles quadratiques auxiliaires $\{F_\kappa\}_{\kappa>0}$ sur un domaine rectangulaire. Sur la base de la famille $\{u_\kappa\}_{\kappa>0}$ on construit une famille de fonctions v_κ telles que $F_\kappa[v_\kappa] \leqslant 2\pi$.

On trouve alors des solutions explicites w_κ d'un système de EDP linéaires de Euler–Langrange qui corespond à la fonctionnelle F_κ, et on montre que $F_\kappa[w_\kappa] > 2\pi$. Ce qui mène à l'inégalité $F_\kappa[v_\kappa] \geqslant F_\kappa[w_\kappa] > 2\pi$ et achève la preuve par l'absurde.

1. Introduction

The present Note establishes nonexistence of minimizers of the Ginzburg–Landau functional in a class of Sobolev functions with prescribed degree on the boundary of an annular domain when the H^1-capacity of the domain is less than the critical value $c_{\mathrm{cr}} = \pi$. Here an annular domain is any domain in $\mathbb{R}^2$ conformally equivalent to a circular annulus.

Consider the minimization problem for the Ginzburg–Landau functional

$$E_\kappa[u] = \frac{1}{2}\int_A |\nabla u|^2\,\mathrm{d}x + \frac{\kappa^2}{4}\int_A \big(|u|^2 - 1\big)^2\,\mathrm{d}x \to \inf, \quad u \in \mathcal{J}, \tag{1}$$

where $A = \Omega \setminus \overline{\omega}$, $\overline{\omega} \subset \Omega$, and ω, Ω are bounded, simply connected domains in $\mathbb{R}^2$ with smooth boundaries. The class $\mathcal{J}$ is defined by

$$\mathcal{J} = \big\{u \in H^1(A)\colon\ |u| = 1 \text{ on } \partial\Omega \cup \partial\omega;\ \deg(u, \partial\Omega) = \deg(u, \partial\omega) = 1\big\}. \tag{2}$$

Note that a minimizer of (1) in $\mathcal{J}$ satisfies the Ginzburg–Landau equation

$$-\Delta u + \kappa^2\big(|u|^2 - 1\big)u = 0 \quad \text{in } A, \tag{3}$$

along with the natural boundary conditions $\frac{\partial u}{\partial \nu} \times u = 0$ on ∂A.

Problem (1) originates with a Ginzburg–Landau variational model of superconducting persistent currents in multiply connected domains.

The asymptotics as $\kappa \to \infty$ of global minimizers for the Ginzburg–Landau functional and their vortex structure for the Dirichlet boundary data (for which the degree is fixed by default) were studied in detail in [8] for simply-connected domains. For multiply connected domains and the Ginzburg–Landau functional with a magnetic field, the existence of local minimizers for large κ was established in [11] (cf. [10]) while the existence and properties of *global* minimizers was studied in [2].

The variational problems for the Ginzburg–Landau functional in a class of maps with the degree boundary conditions were considered in [7,9,4–6]. The difficulty in establishing the existence of minimizers over the class $\mathcal{J}$ is due to the fact that $\mathcal{J}$ is not closed with respect to weak H^1-topology [6] and one cannot use the direct method of calculus of variations. For a narrow circular annulus both existence and uniqueness were proved in [9] for an *arbitrary* (not necessarily large) $\kappa > 0$. In [4–6], a general approach for arbitrary multiply connected domains was developed. The existence of critical H^1-capacity, $c_{\mathrm{cr}} = \pi$, was established—the minimizers were shown always to exist when a domain has the capacity $\operatorname{cap}(A)$ that exceeds π. When $\operatorname{cap}(A)$ is below π, the minimizers of E_κ were shown to exist only when κ is small. Further it was conjectured in [4–6] that, when $\operatorname{cap}(A) < \pi$, there exists a threshold value κ_1 such that the minimum of E_κ is *not attained* when $\kappa > \kappa_1$ and it is attained when $\kappa < \kappa_1$.

The existence of κ_1 is established in this Note via the following:

Theorem 1.1. *Let $m_\kappa = \operatorname{Inf}\{E_\kappa[u],\ u \in \mathcal{J}\}$. Assume $\operatorname{cap}(A) < \pi$. Then there is a finite $\kappa_1 > 0$ such that m_κ is always attained for $\kappa < \kappa_1$ and it is never attained for $\kappa > \kappa_1$.*