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Partial Differential Equations

Magnetic Ginzburg–Landau functional with discontinuous constraint

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Abstract

This Note reports on results obtained for minimizers of a Ginzburg–Landau functional with discontinuous constraint. These results concern vortex-pinning and boundary conditions for inhomogeneous superconducting samples. ***To cite this article: A. Kachmar, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

Résumé

Une fonctionnelle de Ginzburg–Landau magnétique avec une contrainte discontinue. Cette Note rend compte sur des résultats récents obtenus pour les minimiseurs d'une fonctionnelle de Ginzburg–Landau avec une contrainte discontinue. Ces résultats concernent le phénomène de chevillage (pinning) de vortex et les conditions aux limites pour des échantillons supraconducteurs inhomogènes. ***Pour citer cet article : A. Kachmar, C. R. Acad. Sci. Paris, Ser. I 346 (2008).***

1. Ginzburg–Landau functional with discontinuous constraint

Some physical experiments deal with superconducting samples subject to non-constant temperatures, or with samples consisting of superconducting materials with different critical temperatures; see [10] for a recent review concerning these experiments. Such superconducting samples are of particular interest since they permit to increase (or decrease) the value of the onset field H_{C_3} (third critical field), and they serve in controlling the position of vortices, exhibiting thus a phenomenon known as *vortex-pinning*.

In the framework of the Ginzburg–Landau theory, it is proposed to model the energy of an inhomogeneous superconducting sample by means of the following functional (see [2]):

$$\mathcal{G}_{\varepsilon,H}(\psi,A)=\int_{\Omega}\left(\left|(\nabla-\mathrm{i}A)\psi\right|^2+\frac{1}{2\varepsilon^2}\left(p(x)-|\psi|^2\right)^2+|\operatorname{curl}A-H|^2\right)\mathrm{d}x. \tag{1}$$

Here $\Omega\subset\mathbb{R}^2$ is the 2-D cross section of the sample (assumed to occupy a cylinder of infinite height), $H\geqslant 0$ is the intensity of the applied magnetic field, $\frac{1}{\varepsilon}=\kappa>0$ is the Ginzburg–Landau parameter (a temperature independent

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parameter), and $p(x)$ is a real valued function whose values are determined by the local temperature in the sample. The functional (1) is defined for pairs (ψ, A) in the space $\mathcal{H} = H^1(\Omega;\mathbb{C}) \times H^1(\Omega;\mathbb{R}^2)$.

Lassoued and Mironescu [9] analyzed the functional (1) without a magnetic field (i.e. $H = 0$ and $A = 0$) and when the function $p(x)$ is a step function. Aftalion, Sandier and Serfaty [1] analyzed the functional (1) when the function $p(x)$ is smooth and strictly positive. We analyze the functional (1) in the following case:

$$\Omega = B(0,1), \quad p(x) = 1 \quad \text{in } B(0,R), \quad p(x) = a \quad \text{in } B(0,1) \setminus B(0,R), \tag{2}$$

where $B(0,1)$ denotes the unit disc in $\mathbb{R}^2$, $R \in]0,1[$ and $a \in \mathbb{R}_+ \setminus \{1\}$ are given constants.

2. The case without a magnetic field

The next theorem characterizes the set of minimizers of the functional (1) when there is no applied magnetic field, i.e. $H = 0$.

Theorem 1. *Assume that $H = 0$. Up to a gauge transformation, the functional* (1) *admits in the space $\mathcal{H}$ a unique minimizer $(u_\varepsilon, 0)$, where $u_\varepsilon : \Omega \to \mathbb{R}$ is a non-negative function.*

Moreover, there exists a constant ε_0 such that for all $\varepsilon \in]0, \varepsilon_0[$, $u_\varepsilon \in C^2(\overline{B(0,R)}) \cup C^2(\overline{B(0,1) \setminus B(0,R)})$ and

$$\min(1, \sqrt{a}) < u_\varepsilon < \max(1, \sqrt{a}) \quad \text{in } \overline{\Omega}.$$

To understand the asymptotic behavior of the function u_ε, we show that there is a unique positive and bounded function $U : \mathbb{R} \to \mathbb{R}$ that solves the equation:

$$-U''(t) = \big(p_0(t) - U^2(t)\big)U(t) \quad \text{in } \mathbb{R}, \quad \text{where } p_0(t) = 1 \text{ in } \mathbb{R}_- \text{ and } p_0(t) = a \text{ in } \mathbb{R}_+. \tag{3}$$

The expression of $U(t)$ can be given explicitly (see [5]), but we only need to know that the quantity $\gamma(a) = U'(0)/U(0)$ is positive when $a < 1$, and negative when $a > 1$.

By a blow-up argument, we are able to describe the asymptotic behavior of the function u_ε by means of the one-dimensional function U.

Theorem 2. *The following asymptotic limits hold as $\varepsilon \to 0$:*

$$\lim_{\varepsilon\to 0} \left\| u_\varepsilon(x) - U\left(\frac{|x| - R}{\varepsilon}\right) \right\|_{L^\infty(\Omega)} = 0, \tag{4}$$

$$\forall C > 0, \quad \lim_{\varepsilon\to 0} \varepsilon \left\| u_\varepsilon(x) - U\left(\frac{|x| - R}{\varepsilon}\right) \right\|_{W^{1,\infty}(\{x\in\mathbb{R}^2:\ |R - |x|| \leqslant C\varepsilon\})} = 0. \tag{5}$$

3. Vortex pinning

We return to the analysis of minimizers of the functional (1) in the presence of an applied magnetic field, i.e. $H > 0$.

Theorem 3. *Let $(\psi_{\varepsilon,H}, A_{\varepsilon,H})$ be a minimizer of* (1). *There exists a constant $a_0 \in]0,1[$, and for each $a \in]0, a_0[$, there exist positive constants μ_*, $\mu_\#$, ε_0 and a function $]0, \varepsilon_0[\ni \varepsilon \mapsto k_\varepsilon \in \mathbb{R}_+$, $0 < \liminf_{\varepsilon\to 0} k_\varepsilon \leqslant \limsup_{\varepsilon\to 0} k_\varepsilon < \infty$, such that*:

(i) *If $H < k_\varepsilon|\ln\varepsilon| - \mu_* \ln|\ln\varepsilon|$, then $|\psi_{\varepsilon,H}| \geqslant \frac{\sqrt{a}}{2}$ in $\overline{\Omega}$.*

(ii) *If $H = k_\varepsilon|\ln\varepsilon| + \mu \ln|\ln\varepsilon|$ and $\mu \geqslant -\mu_*$, then there exists a finite family of balls $(B(a_i(\varepsilon), r_i(\varepsilon)))_i$ with the following properties*:
  (a) $\sum_i r_i(\varepsilon) < |\ln\varepsilon|^{-10}$;
  (b) $|\psi_{\varepsilon,H}| \geqslant \frac{\sqrt{a}}{2}$ *in* $\overline{\Omega} \setminus \bigcup_i B(a_i(\varepsilon), r_i(\varepsilon))$;
  (c) *Letting d_i be the degree of $\psi_{\varepsilon,H}/|\psi_{\varepsilon,H}|$ on $\partial B(a_i(\varepsilon), r_i(\varepsilon))$ if $B(a_i, r_i) \subset \Omega$ and 0 otherwise, then we have*

$$\sup_{i,|d_i|>0} \big|R - |a_i(\varepsilon)|\big| \to 0 \quad \text{as } \varepsilon \to 0.$$