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Schwarz lemma for pluriharmonic functions☆

Zhenghua Xu

Department of Mathematics, University of Science and Technology of China, Hefei 230026, China

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Abstract

In this note the Schwarz theory for pluriharmonic functions is studied, including the Schwarz lemma, the Julia lemma, and the behavior of invariant metric for pluriharmonic functions in the unit ball.

Keywords: Pluriharmonic functions; Schwarz lemma; Bergman metric; Julia lemma

1. Introduction

The Schwarz lemma for holomorphic functions is always an amazing topic in complex analysis [1,3,4]. Recently, some interesting works are focused on its extensions to planar harmonic functions [2,7–9] as well as vector-valued holomorphic functions [10]. For instance, Kalaj and Vuorinen [7] found that the Bergman metric in the unit disc B^1 is decreasing up to a constant under any harmonic function $u : B^1 \longrightarrow (-1, 1)$. In this article we initiate to extend the theory to higher dimension for pluriharmonic functions.

Let B^n be the open unit ball in $\mathbb{C}^n$. A real-valued function u defined on a domain $\Omega \subset \mathbb{C}^n$ is said to be pluriharmonic if, for each fixed $z \in \Omega$ and $\theta \in \partial B^n$, the function $u(z + \theta\zeta)$ is harmonic in the complex variable ζ, for $|\zeta|$ smaller than the distance of z from $\partial\Omega$. It is known that a function u of B^n into $\mathbb{R}$ is pluriharmonic if and only if u is the real part of a holomorphic

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E-mail address: xzhengh@mail.ustc.edu.cn.

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function [11]. Denote

$$\nabla u(z) = \left(\frac{\partial}{\partial x_1} + i \frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial x_n} + i \frac{\partial}{\partial y_n} \right) u(z),$$

the gradient of u at $z = (x_1 + iy_1, \ldots, x_n + iy_n) \in B^n$, and denote $e_1 = (1, 0, \ldots, 0) \in \partial B^n$.

Now we can state our main results. We first extend the result of Kalaj and Vuorinen [7] on the Schwarz lemma for planar harmonic functions to pluriharmonic functions.

Theorem 1.1. *Let $u : B^n \longrightarrow (-1, 1)$ be pluriharmonic. Then for any $z, w \in B^n$ we have*

$$k_{B^n}(u(z), u(w)) \leq \frac{4}{\pi} k_{B^n}(z, w),$$

where $k_{B^n}(z, w)$ stands for the Bergman distance between $z, w \in B^n$.

Theorem 1.1 implies a boundary version of the Schwarz lemma for pluriharmonic functions, which extends the result of Chen [2] for planar harmonic mappings to higher dimensions.

Theorem 1.2. *Let $u : B^n \cup \{e_1\} \longrightarrow \mathbb{R}$ be pluriharmonic such that $u(B^n) \subset (-1, 1)$ and $u(e_1) = 1$. Then*

$$a := |\nabla u(e_1)| = \frac{\partial u}{\partial x_1}(e_1) > 0$$

and

$$u(z) \geq \frac{4}{\pi} \arctan \frac{1 - \sigma}{1 + \sigma}, \quad \forall z \in B^n,$$

where

$$\sigma = \frac{\pi a}{2} \frac{|1 - (z, e_1)|^2}{1 - |z|^2}.$$

We remark that the holomorphic version of Theorem 1.2 is the classical Julia lemma [1,5] which can be considered as a boundary Schwarz lemma. It states that if $f : B^1 \cup \{1\} \longrightarrow B^1 \cup \{1\}$ be holomorphic with $f(1) = 1$, then $f'(1) > 0$ and

$$\frac{|1 - f(z)|^2}{1 - |f(z)|^2} \leq f'(1) \frac{|1 - z|^2}{1 - |z|^2} =: \sigma_1, \quad \forall z \in B^1.$$

In particular,

$$\operatorname{Re} f(z) \geq \frac{1 - \sigma_1}{1 + \sigma_1}, \quad \forall z \in B^1.$$

As a direct consequence of Theorem 1.2, we get a Julia lemma for gradient of pluriharmonic functions.

Theorem 1.3. *Let u be a real pluriharmonic function on $B^n \cup \{e_1\}$ such that $u(B^n) \subset (-1, 1)$, $u(0) = 0$ and $u(e_1) = 1$. Then*

$$|\nabla u(e_1)| \geq \frac{2}{\pi}.$$

Moreover, the inequality is sharp and the equality holds for $u(z) = \frac{4}{\pi} \operatorname{Re}(\arctan(z, e_1))$.