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Zeros of quasi-orthogonal ultraspherical polynomials

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Abstract

For each fixed value of λ in the range $-3/2 < \lambda < -1/2$, we prove interlacing properties for the zeros of polynomials, of consecutive and non-consecutive degree, within the sequence of quasi-orthogonal order 2 ultraspherical polynomials $\{C_n^{(\lambda)}\}_{n=0}^{\infty}$. We investigate the manner in which interlacing occurs between the zeros of quasi-orthogonal order 2 ultraspherical polynomials $C_n^{(\lambda)}$ and their orthogonal counterparts $C_m^{(\lambda+1)}$ and derive necessary and sufficient conditions for interlacing to occur between the zeros of $C_n^{(\lambda)}$ and $C_m^{(\lambda+2)}$, $n, m \in \mathbb{N}$, $-3/2 < \lambda < -1/2$. In some cases we get interlacing when we include additional factors such as x, $1 - x^2$, and occasionally more complicated factors to the polynomials being considered.

Keywords: Ultraspherical polynomials; Zeros; Interlacing

1. Introduction

Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials that is orthogonal with respect to a positive Borel measure μ supported on a finite or infinite interval (a, b). It is well known from general properties of orthogonal polynomials (see, e.g., [22, Theorem 3.3.1]) that the zeros of p_n are real and simple

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and lie in the interval of orthogonality (a, b) while, if we denote the zeros of $p_n, n \in \mathbb{N}$, in increasing order by $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$, then [22, Theorem 3.3.2]

$$x_{1,n} < x_{1,n-1} < x_{2,n} < x_{2,n-1} < \cdots < x_{n-1,n-1} < x_{n,n}, \tag{1}$$

a property called the interlacing of zeros. Since our discussion will include interlacing of zeros of polynomials of the same degree, and of consecutive and non-consecutive degrees, we recall the following definitions.

Definition 1.1. Let $n \in \mathbb{N}$. If $x_{1,n} < x_{2,n} < \cdots < x_{n,n}$ are the zeros of p_n and $y_{1,n} < y_{2,n} < \cdots < y_{n,n}$ are the zeros of q_n, then the zeros of p_n and q_n are interlacing if

$$x_{1,n} < y_{1,n} < x_{2,n} < y_{2,n} < \cdots < x_{n,n} < y_{n,n} \tag{2}$$

or if

$$y_{1,n} < x_{1,n} < y_{2,n} < x_{2,n} < \cdots < y_{n,n} < x_{n,n}. \tag{3}$$

In case p_n is replaced by p_{n+1}, (2) is replaced by

$$x_{1,n+1} < y_{1,n} < x_{2,n+1} < y_{2,n} < \cdots < x_{n,n+1} < y_{n,n} < x_{n+1,n+1}. \tag{4}$$

The interlacing of zeros of two polynomials whose degrees differ by two or more was introduced by Stieltjes. See [22, Section 3.3].

Definition 1.2. Let $m, n \in \mathbb{N}$, $m \le n-1$. The zeros of the polynomials p_n and q_m are interlacing if there exist m open intervals, with endpoints at successive zeros of p_n, each of which contains exactly one zero of q_m.

The interlacing property of zeros of polynomials of consecutive degree in a sequence is useful in providing relatively straightforward proofs of quadrature rules; see [16].

The concept of quasi-orthogonality of order 1 was introduced by Riesz [20] in connection with his work on the moment problem. Fejér [15] considered quasi-orthogonality of order 2 and the general case was studied by Shohat [21] and many other authors including Chihara [6], Dickinson [7], Draux [8] and Maroni [17–19]. In recent papers [4,5,2], it is proved that the highest possible degree of accuracy of an n-point quadrature formula with one or two nodes fixed is achieved when the remaining nodes of the quadrature rule are the zeros of R_n where the sequence $\{R_n\}_{n=0}^{\infty}$ is quasi-orthogonal with respect to a positive Borel measure μ. The definition of quasi-orthogonality of a sequence is the following:

Definition 1.3. Let $\{q_n\}_{n=0}^{\infty}$ be a sequence of polynomials with degree $q_n = n$ for each $n \in \mathbb{N}$. For a positive integer $r < n$, the sequence $\{q_n\}_{n=0}^{\infty}$ is quasi-orthogonal of order r with respect to a positive Borel measure μ if

$$\int x^k q_n(x)\, d\mu(x) = 0 \quad \text{for } k = 0, \ldots, n-1-r. \tag{5}$$

If (5) holds for $r = 0$, the sequence $\{q_n\}_{n=0}^{\infty}$ is orthogonal with respect to the measure μ.