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On the commutation of generalized means on probability spaces

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Abstract

Let f and g be real-valued continuous injections defined on a non-empty real interval I, and let $(X, \mathscr{L}, \lambda)$ and $(Y, \mathscr{M}, \mu)$ be probability spaces in each of which there is at least one measurable set whose measure is strictly between 0 and 1.

We say that (f, g) is a (λ, μ)-switch if, for every $\mathscr{L} \otimes \mathscr{M}$-measurable function $h : X \times Y \to \mathbf{R}$ for which $h[X \times Y]$ is contained in a compact subset of I, it holds

$$f^{-1}\left(\int_X f\left(g^{-1}\left(\int_Y g \circ h \, d\mu\right)\right) d\lambda\right) = g^{-1}\left(\int_Y g\left(f^{-1}\left(\int_X f \circ h \, d\lambda\right)\right) d\mu\right),$$

where f^{-1} is the inverse of the corestriction of f to $f[I]$, and similarly for g^{-1}.

We prove that this notion is well-defined, by establishing that the above functional equation is well-posed (the equation can be interpreted as a permutation of generalized means and raised as a problem in

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the theory of decision making under uncertainty), and show that (f, g) is a (λ, μ)-switch if and only if $f = ag + b$ for some $a, b \in \mathbf{R}$, $a \neq 0$.

Keywords: Commuting mappings; Functional equations; Generalized [quasi-arithmetic] means; Permutable functions

1. Introduction

Below, we let $I \subseteq \mathbf{R}$ be a non-empty interval, which may be bounded or unbounded, and neither open nor closed. We will need the following proposition, which is proved in Section 3 (see Section 2 for a glossary of notation and terms used but not defined in this introduction):

Proposition 1. *Let $(S, \mathscr{C}, \gamma)$ be a probability space, and assume $w : I \to \mathbf{R}$ and $h : S \to I$ are functions such that $w[I]$ is an interval and $w \circ h$ is γ-integrable. Then $\int_S w \circ h \, d\gamma \in w[I]$.*

Given $(S, \mathscr{C}, \gamma)$ and w as in Proposition 1, we denote by $\mathfrak{L}^w(\gamma)$ the set of all $\mathscr{C}$-measurable functions $h : S \to I$ such that $w \circ h$ is γ-integrable, while we write $\mathcal{H}(\gamma)$ for the set of all $\mathscr{C}$-measurable functions $h : S \to I$ for which $h[S] \Subset I$.

Based on these premises, assume w is an injection, so that we can consider the inverse, w^{-1}, of w. It follows from Proposition 1 that the functional

$$\mathfrak{L}^w(\gamma) \to \mathbf{R} : h \mapsto w^{-1}\left(\int_S w \circ h \, d\gamma\right), \tag{1}$$

which we denote by $\mathfrak{F}_\gamma(w)$ and refer to as the *w-mean relative to γ*, is well-defined and its image is contained in I. For $h \in \mathfrak{L}^w(\gamma)$ we call $\mathfrak{F}_\gamma(w)(h)$ the w-mean of h relative to γ.

The naming comes from the observation that, if I is the interval $]0, \infty[$ and w is, for some real $p \neq 0$, the function $I \to \mathbf{R} : x \mapsto x^p$, then $\mathfrak{L}^w(\gamma)$ is the set of all ($\mathscr{C}$-measurable and positive) functions $S \to I$ whose pth power is γ-integrable, while $\mathfrak{F}_\gamma(w)$ is the integral mean

$$\mathfrak{L}^w(\gamma) \to \mathbf{R} : h \mapsto \left(\int_S h^p \, d\gamma\right)^{\frac{1}{p}}.$$

When S is a finite set, (1) gives a generalization of classical and weighted means (say, the arithmetic mean, the quadratic mean, the harmonic mean, and others) first considered, respectively, by A. Kolmogorov and M. Nagumo [15,22] and B. de Finetti and T. Kitagawa [9,14].

Indeed, our interest in Proposition 1 is mainly due to the following result, which also will be proved in Section 3.

Proposition 2. *Let $(U, \mathscr{A}, \alpha)$ be a measure space and $(V, \mathscr{B}, \beta)$ a probability space, and let w be a continuous injection $I \to \mathbf{R}$ and h a function $U \times V \to I$. The following hold:*

(i) *Let $w \circ h_x$ be β-integrable for every $x \in U$, where h_x is the map $V \to \mathbf{R} : y \mapsto h(x, y)$. Then the function $\varphi : U \to \mathbf{R} : x \mapsto \mathfrak{F}_\beta(w)(h_x)$ is well-defined and $\varphi[U] \subseteq I$. Moreover, if h is $\mathscr{A} \otimes \mathscr{B}$-measurable and $w \circ h$ is bounded, then φ is $\mathscr{A}$-measurable.*

(ii) *Suppose that $h[U \times V] \Subset I$, and let h be $\mathscr{A} \otimes \mathscr{B}$-measurable. Then $\varphi[U] \subseteq I$, and φ is $\mathscr{A}$-measurable and bounded.*