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Complete connections on fiber bundles

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Abstract

Every smooth fiber bundle admits a complete (Ehresmann) connection. This result appears in several references, with a proof on which we have found a gap, that does not seem possible to remedy. In this note we provide a definite proof for this fact, explain the problem with the previous one, and illustrate with examples. We also establish a version of the theorem involving Riemannian submersions.

Keywords: Ehresmann connections; Fiber bundles; Riemannian submersions

1. Introduction: A rather tricky exercise

An *(Ehresmann) connection* on a submersion $p : E \to B$ is a smooth distribution $H \subset TE$ that is complementary to the kernel of the differential, namely $TE = H \oplus \ker \mathrm{d}p$. The distributions H and $\ker \mathrm{d}p$ are called *horizontal* and *vertical*, respectively, and a curve on E is called horizontal (resp. vertical) if its speed only takes values in H (resp. $\ker \mathrm{d}p$). Every submersion admits a connection: we can take for instance a Riemannian metric η^E on E and set H as the distribution orthogonal to the fibers.

Given $p : E \to B$ a submersion and $H \subset TE$ a connection, a smooth curve $\gamma : I \to B$, $t_0 \in I$, locally defines a *horizontal lift* $\tilde{\gamma}_e : J \to E$, $t_0 \in J \subset I$, $\tilde{\gamma}_e(t_0) = e$, for e an arbitrary point in the fiber. This lift is unique if we require J to be maximal, and depends smoothly on e.

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The connection H is said to be *complete* if for every γ its horizontal lifts can be defined in the whole domain. In that case, a curve γ induces diffeomorphisms between the fibers by *parallel transport*. See e.g. [9] for further details.

The purpose of this article is to show that, when B is connected, a submersion $p : E \to B$ admits a complete connection if and only if p is a *fiber bundle*, namely if there are *local trivializations* $\phi_i : p^{-1}(U_i) \to U_i \times F, \pi_1\phi_i = p$. One implication is easy: if H is a complete connection, working locally, we can assume U_i is a ball in $\mathbb{R}^n$, and define $\phi_i^{-1} : U_i \times p^{-1}(0) \to p^{-1}(U_i)$ by performing parallel transport along radial segments, obtaining a fiber bundle over each component of B. The converse, as we shall see, is definitely more challenging.

As far as we know, this result first appeared in [10, Cor 2.5], with a proof that turned out to be incorrect, and then as an exercise in [4, Ex VII.12]. Later it was presented as a theorem in [6–9], always relying in a second proof, that P. Michor attributed to S. Halperin in [7], and that we learnt from [2]. We have found a gap in that argument, that does not seem possible to remedy. Concretely, it is assumed that fibered metrics are closed under convex combinations. A counterexample for this can be found in [1, Ex 2.1.3].

In Section 2 we prove that every fiber bundle admits a complete connection. Our strategy uses local complete connections and a partition of 1, as done by Michor, but we allow our coefficients to vary along the fibers. We do it in a way so as to make the averaged connection and the local ones to agree in enough horizontal sections, which we show insures completeness. In Section 3 we discuss fibered complete Riemannian metrics, provide counter-examples to some constructions in the literature, and show that every fiber bundle admits a complete fibered metric, concluding the triple equivalence originally proposed in [10].

2. Our construction of complete connections

Given (U_i, ϕ_i) a local trivialization of $p : E \to B$, there is an *induced connection* on $p : p^{-1}(U_i) \to U_i$ defined by $H_i = \mathrm{d}\phi_i^{-1}(TU_i \times 0_F)$, and is complete. The space of connections inherits a convex structure by identifying each connection H with the corresponding projection onto the vertical component. It is tempting then to construct a global complete connection, out of the ones induced by trivializations, by using a partition of 1. The problem is that, as stated in [4], complete connections are not closed under convex combinations.

Example 1. Let $p : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate, and let H_1, H_2 be the connections spanned by the following horizontal vector fields:

$$H_1 = \langle \partial_x + 2y^2 \sin^2(y)\partial_y \rangle \qquad H_2 = \langle \partial_x + 2y^2 \cos^2(y)\partial_y \rangle.$$

Note that the curves $t \mapsto (t, k\pi), k \in \mathbb{Z}$, integrate H_1, and because of them, any other horizontal lift of H_1 is bounded and cannot go to ∞. The same argument applies to H_2. Hence both connections are complete. However, the averaged connection $\frac{1}{2}(H_1 + H_2)$ is spanned by the horizontal vector field $\partial_x + y^2\partial_y$ and is not complete.

Our strategy to prove that every fiber bundle $p : E \to B$ admits a complete connection is inspired by previous example. We will paste the connections induced by local trivializations by using a partition of 1, in a way so as to preserve enough local horizontal sections, that will bound any other horizontal lift of a curve. Given $U \subset B$ an open, we say that a local section $\sigma : U \to E$ is *horizontal* if $\mathrm{d}\sigma$ takes values in H, and we say that a family of local sections $\{\sigma_k : U \to E\}_k$ is *disconnecting* if each component of $p^{-1}(U) \setminus \bigcup_k \sigma_k(U)$ has compact closure in E.