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An application of Poénaru's "zipping theory"

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Abstract

In this note we present an application of the "zipping theory" introduced by V. Poénaru in the 80s, aimed to kill in a controlled way all the singularities of a non-degenerate simplicial map $f : X \to M$, from a simplicial complex to a manifold. By means of some elementary *zipping moves*, it is possible to perform a concrete algorithm which produces the 'smallest' equivalence relation on X, compatible with f, and which kills all the singularities of f. We use this zipping process for listing both finite 3-complexes which are homotopy equivalent to the 3-sphere, and finite 3-complexes with a finite fundamental group.

Keywords: 3-manifolds; Singularities; Foldings; Turing machine; Algorithmic problems

1. Introduction

Computational topology is a fascinating area, at the intersection of algebraic topology, group theory, complexity theory and computer science. Probably, the proof by Dehn and Heegaard of the surface classification theorem, and Dehn's algorithm for the word problem in finitely presented groups, may be considered the very first algorithmic-type results in topology; while one of the first analysis about the efficiency of a topological algorithm goes back to Haken's famous test of the triviality of a knot [1,4,6]. On the other hand, it is known that several algorithmic problems about manifolds are undecidable [19]: for instance, Markov proved that determining whether two manifolds of dimension 4 with a simplicial structure are homeomorphic

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is undecidable (this follows from the classical result of Novikov showing that there exists a finitely presented group for which the word problem is undecidable). On the contrary, Whittlesey showed that classifying finite 2-complexes up to homeomorphism is possible, so that the main area of interest is actually in dimension 3, in particular in order to understand the complexity of deciding if two 3-manifolds are homeomorphic: this is a much more delicate issue. For precise references and details see e.g. [1,3–8,17–19].

The present note fits in this research field since it presents an application of some abstract topological techniques developed by V. Poénaru — the so-called *zipping process* or Ψ/Φ-theory, introduced in [11] and used in [9,10,12–16] – to some algorithmic problems in dimension 3. Although the results we prove in this paper could possibly be known by experts, the algorithms we use are first of all new, to the best of our knowledge, and also, presumably, quick, simple and efficient. Furthermore, our proofs are of topological nature, direct and short.

The Ψ/Φ-machinery is a tool which comes in help when one deals with singularities of simplicial maps, and it works by means of successive "zipping" moves (see e.g. [2] or [9,11, 13]), and this zipping process is, in some sense, elementary and very fast [15]. Hence, there is a reasonable hope that our techniques might be able to tackle, from a different and new perspective, some of the questions in the field.

Before stating our results, we need to recall several technical preliminaries from [11,13].

1.1. *Preliminaries on the zipping process*

Let us introduce briefly Poénaru's *zipping theory*, namely the Ψ/Φ-technology of [11]. To begin with, recall that, just like a closed surface can be represented by identifying the edges of a polygon, any closed n-dimensional PL-manifold M^n may be obtained by identifying opportunely the sides of a n-dimensional polyhedral ball Δ^n. By unrolling this fundamental domain Δ^n along its faces, one gets an infinite *arborescent* union T of copies of Δ^n, which may be non locally-finite, and which is naturally equipped with a non-degenerate map $f : T \to M^n$. The map f is simply the tautological map which sends each fundamental domain $\Delta^n_j \subset T$ identically onto $\Delta^n \to M^n$ (in other words, f unrolls indefinitely the fundamental domain $\Delta^n \to M^n$ along its faces). Of course, by construction, this tree-like object T has lots of *singularities* (i.e. points where T is not an n-manifold), and the zipping-process is actually a practical strategy for getting rid of them, but still preserving some useful topological information (for more details on this construction see [12,15,16] and Section 2.1 of the present note).

Motivated by this way of representing closed manifolds, Poénaru considered the very general situation of a non-degenerate simplicial map $f : X \to M^n$, where M^n is a closed (triangulated) n-manifold (or a locally-finite simplicial complex) and X a countable Gromov *multicomplex* (a generalization of simplicial complexes, where the intersection of two simplexes is not just a common face, but a subcomplex) which is not necessarily locally-finite.

Remark 1.1. Observe that, since the simplicial map f is *non-degenerate* (which means that the image of a simplex is a simplex of the same dimension), we will always have $\dim X \leq \dim M$.

Usually, one restricts to low-dimensions (2 and 3) for the space X because in such a case the zipping process is most transparent, effective and handy.

The map f comes with two specific sets: the set of *double points* of f, $M^2(f) = \{(x, y) \subset X \times X$, with $x \neq y,$ such that $f(x) = f(y)\}$, and the set (actually the subcomplex) of *singularities* of f, $\mathrm{Sing}(f) \subset X$, namely the set of points $x \in X$ such that $f|_{Star(x)}$ is not injective. In other words, $x \in \mathrm{Sing}(f)$ if and only if there are two distinct simplexes of the same dimension