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Classification of eventually periodic subshifts

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Abstract

We provide a classification of eventually periodic subshifts up to conjugacy and flow equivalence. We use our results to prove that each skew Sturmian subshift is conjugate to exactly one other skew Sturmian subshift and that all skew Sturmian subshifts are flow equivalent to one another.

Keywords: Skew Sturmian sequences; Conjugacy; Flow equivalence

0. Introduction

In this paper we study non-periodic bi-infinite sequences of symbols that, when some finite word is removed, yield periodic bi-infinite sequences. We term these eventually periodic sequences and define an anomaly word as the word that, when removed, leaves the sequence

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periodic. By considering the closure of the orbit of a given bi-infinite sequence, we obtain a subshift, which is a symbolic dynamical system. It is customary to name a subshift after the sequence from which it is created; in this fashion, we refer to periodic subshifts, eventually periodic subshifts, and so on. While the classification of periodic subshifts is well understood and even straightforward, eventually periodic subshifts are strictly sofic, so their classification is somewhat more elaborate. We prove that the conjugacy class of an eventually periodic subshift depends only on the size of the anomaly and the least period of its corresponding periodic sequence and also that, as is the case for periodic subshifts, there is a single flow equivalence class of eventually periodic subshifts.

Motivation for this work was given by the skew Sturmian sequences introduced in [13], which are a particular class of eventually periodic sequences. Sturmian sequences, especially those which are not periodic nor eventually periodic, have attracted the attention of many mathematicians [2,4–6,10,12]. However, skew Sturmian sequences are important to study since they appear naturally in the generalization of properties of Sturmian sequences [1,3,7,16]. By showing in this paper how to compute the size of an anomaly word of a skew Sturmian sequence, we are able to apply our results to obtain a characterization for conjugacy of skew Sturmian subshifts. We show that all skew Sturmian subshifts have a conjugacy class of size two, regardless of their least period. In contrast, we also show that all skew Sturmian subshifts belong to the same flow equivalence class, since they are all eventually periodic.

We divide this work into three sections. In Section 1 we introduce the necessary notions and notations required in the paper. In Section 2 we state the classification results concerning conjugacy and flow equivalence of eventually periodic subshifts. Finally, in Section 3 we apply our results to classify skew Sturmian subshifts.

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1. Preliminaries

In this section we introduce the necessary terminology used in the paper. For the symbolic dynamics notation and basic results we follow [11]. Throughout this paper $\mathcal{A}$ denotes a finite set, called an alphabet, and its elements are referred as its symbols or letters. The space $\mathcal{A}^{\mathbb{Z}}$ of all bi-infinite sequences in $\mathcal{A}$ is a metric space referred to as the full shift. We view elements in $\mathcal{A}$ as sequences $x = (x_i)$ with $i \in \mathbb{Z}$ and $x_i \in \mathcal{A}$. By a word (or block) of length n we mean a finite sequence of n symbols of $\mathcal{A}^{\mathbb{Z}}$. It is customary to denote by $|w|$ the length of a word w. Given $x \in \mathcal{A}^{\mathbb{Z}}$ we say that a word w occurs (or is allowed) in x if there is an $i \in \mathbb{Z}$ and an $n \geq 0$ such that $w = x_i x_{i+1} \dots x_{i+n} = x_{[i,i+n]}$. In addition, we say that a word w is forbidden in x if it does not occur in x. Naturally, a subword of a word w is a word itself that occurs in w.

The shift map $\sigma\colon \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ defined by $\sigma(x)_i = x_{i+1}$ for $i \in \mathbb{Z}$ is a homeomorphism implementing a natural action of $\mathbb{Z}$ on $\mathcal{A}^{\mathbb{Z}}$. A subshift or shift space X is a subset of $\mathcal{A}^{\mathbb{Z}}$ that is both closed and σ-invariant. Equivalently, a subset X of the full shift is a subshift if there is a set of words $\mathcal{F}$ such that X consists of every bi-infinite sequence for which the words in $\mathcal{F}$ are forbidden.

If $x, y \in \mathcal{A}^{\mathbb{Z}}$ and there exists $k \in \mathbb{Z}$ with $\sigma^k(x) = y$, then we say that x and y are similar sequences.

We say that $x \in \mathcal{A}^{\mathbb{Z}}$ is periodic with period $N > 0$ if $\sigma^N(x) = x$. If N is the least integer satisfying this property then we say that N is the least period of x. In particular, when $N = 1$ we