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A non-commutative version of Nikishin's theorem

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Abstract

Let τ be a tracial normal state on a von Neumann algebra, $L^1(\tau)$ be the space of integrable self-adjoint operators, and S be the space of self-adjoint measurable operators. We prove that every positive linear operator from an ordered Banach space to S can be factorized through $L^1(\tau)$.

Keywords: von Neumann algebra; Tracial normal state; Measurable operator; Positive operator; Factorization of operators

In what follows, τ is a faithful normal tracial state on a (finite) von Neumann algebra M of operators in a Hilbert space H. We will denote by M^{pr} the orthomodular lattice of all orthogonal projections in M and by M_* the predual of M.

Recall some definitions and facts of the non-commutative integration (see, e.g., [5, Ch. IX, Section 2]) adapted to the case of finite trace and self-adjoint operators.

A self-adjoint operator x in H with the spectral resolution

$$x = \int_{-\infty}^{+\infty} \lambda \, de_\lambda^x$$

is said to be *affiliated* with M if $e_\lambda^x \in M^{\text{pr}}$ for all $\lambda \in \mathbb{R}$. We will denote by S and S^+ the set of all self-adjoint operators affiliated with M and the subset of positive operators, respectively. For

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any pair $x, y \in S$ the sum $x + y$ and the product yxy are considered in the strong sense (that is, as the closure of the algebraic sum and product, respectively). The formula

$$\tau(x) = \int_0^{+\infty} \lambda \, d\tau(e_\lambda^x)$$

extends τ to S^+ and for any pair $x, y \in S^+$ it holds $\tau(y^{\frac{1}{2}}xy^{\frac{1}{2}}) = \tau(x^{\frac{1}{2}}yx^{\frac{1}{2}})$. The collection of the sets

$$U(\varepsilon, \delta) = \{x \in S: \exists p \in M^{\mathrm{pr}} \text{ such that } \|pxp\| \le \varepsilon \text{ and } \tau(p^\perp) \le \delta\} \quad (\varepsilon, \delta > 0)$$

forms a basis of neighborhoods of zero in *the topology of convergence in measure* which turns S to be a complete metrizable topological vector space. Note that a compatible F-norm N on S can be determined by

$$N(x) = \tau(|x|(1 + |x|)^{-1}).$$

An operator $x \in S$ is said to be τ*-integrable* if

$$\int_{-\infty}^{+\infty} |\lambda| \, d\tau(e_\lambda^x) < \infty.$$

The space of all such operators is endowed with the norm defined by

$$\|x\|_1 = \int_{-\infty}^{+\infty} |\lambda| \, d\tau(e_\lambda^x).$$

In this way we obtain a Banach space over reals, which we will denote by $L^1(\tau)$.

The crucial point of our paper is Lemma 2. That was stated without proof in [6]. Its proof is essentially the same as the first part of the proof of [7, Lemma VI.5.5]. Nevertheless, we include the proof for the sake of completeness.

Lemma 1 (*[7, Lemma VI.5.4 (Ky Fan)]*)**.** *Let K be a compact convex subset of a topological vector space and Γ be a convex set of lower semicontinuous convex mappings $\Phi : K \to (-\infty, +\infty]$. Suppose that for each $\Phi \in \Gamma$ there exists an $\xi \in K$ such that $\Phi(\xi) \le 0$. Then, there exists an element $\xi_0 \in K$ such that $\Phi(\xi_0) \le 0$ for each $\Phi \in \Gamma$.*

Lemma 2 (*[6, Lemma 9]*)**.** *Let C be a convex and bounded subset of S^+. Then for any $\varepsilon > 0$ there exists $p \in M^{pr}$ such that $\sup_{x\in C} \tau(pxp) < \infty$ and $\tau(p^\perp) < \varepsilon$.*

Proof. Take $\varepsilon > 0$ and let $0 < \delta < \frac{1}{4}\varepsilon$. Since C is bounded, we can find $r_\delta > 0$ such that

$$\sup_{x \in C} N(r_\delta^{-1} x) < \delta.$$

Set

$$K = \{a \in M: 0 \le a \le 1, \tau(a) \ge 1 - 2\delta\}.$$

It is easy to see that K is convex and $\sigma(M, M_*)$-compact. For every $x \in C$ define a functional $\Phi_x : K \to (-\infty, +\infty]$ by

$$\Phi_x(a) = \tau(x^{\frac{1}{2}}ax^{\frac{1}{2}}) - r_\delta \; (=\tau(a^{\frac{1}{2}}xa^{\frac{1}{2}}) - r_\delta).$$