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Review

On conjugations of P-homeomorphisms

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Abstract

Let T_f and T_g be P-homeomorphisms of a circle with break point singularities that is, differentiable except in many countable points where the derivative has a jump. Let T_f and T_g have the same irrational rotation number. We provide a sufficient condition for the β-Hölder continuity of a conjugating map between T_f and T_g. Moreover, we provide a sufficient and a necessary condition for the Lipschitz continuity of the conjugating map. And also we provide a sufficient and necessary condition for the C^1-smoothness of the conjugating map between the break equivalent homeomorphisms T_f and T_g.

Keywords: Circle homeomorphism; Ergodicity; Conjugating map; Hölder continuity; Lipschitz continuity; Smooth function

Contents

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1. Introduction

Let $S^1 = \mathbb{R}/\mathbb{Z}$ with clearly defined orientation, metric, Lebesgue measure and the operation of addition be the *unit circle*. Let T be an orientation preserving homeomorphism of the circle S^1 with lift $f : \mathbb{R} \to \mathbb{R}$, f is continuous, strictly increasing and $f(t+1) = f(t) + 1, t \in \mathbb{R}$. The circle homeomorphism T is then defined by $T\tilde{x} = f(x) \bmod 1$, $\tilde{x} \in S^1$, and $\tilde{x} \equiv x + \mathbb{Z}$ with $x \in [0, 1)$. In the following S^1 will be identified with $[0, 1)$ and $\tilde{x} \in S^1$ with $x \in [0, 1)$. The homeomorphism f is called the *lift* of the homeomorphism T and is defined up to an integer term. Below we denote by T_f the circle homeomorphism T with its lift f. The most important arithmetic characteristic of the homeomorphism T_f of the unit circle S^1 is the *rotation number*

$$\rho(T_f) = \lim_{i\to\infty} \frac{f^i(x)}{i} \bmod 1.$$

Here and below, for a given map f, f^i denotes its ith iterate. Denjoy [9] proved that, if the rotation number $\rho = \rho(T_f)$ is irrational and $\log Df$ is of bounded variation, then T_f is conjugate to the pure rotation $T_\rho : \xi \to \xi + \rho$, that is, there exists an essentially unique homeomorphism T_γ such that $T_f = T_\gamma^{-1} \circ T_\rho \circ T_\gamma$. The problem of smoothness of the conjugacy is a classical problem in one-dimensional dynamics. This problem of smoothness of the conjugacy of smooth diffeomorphisms has come to be very well understood (see, for instance, [5,12–18,21, 23]). An important class of circle homeomorphisms is homeomorphisms with break points, or *P-homeomorphisms* for short (see Definition 2.1). Note that, the class P-homeomorphisms was studied for the first time by Herman in the work [12]. Herman extended the classical Denjoy's result for class P-homeomorphisms. The exact statement of the corresponding theorem will be given later (see Theorem 2.4).

In general their ergodic properties such as their invariant measures, their renormalizations and also their rigidity properties are rather different from those of diffeomorphisms (see [1,2,11]).

Next we consider a problem of the regularity of the conjugating map between two P-homeomorphisms with identical irrational rotation number. The case of one break point with the same jump ratios, the so-called rigidity problem, was studied in detail by Khanin and Khmelev in [15] and by Teplinskii and Khanin in the works [22,19,20]. Recently the strongest result in this direction has been obtained by Khanin et al. [16]. It was shown in [16] that for almost all rotation numbers, every two $C^{2+\alpha}$-smooth circle diffeomorphisms with a break point, with the same irrational rotation number and the same jump ratio, are C^1-smoothly conjugate to each other. In the case of non-coinciding jump ratios Dzhalilov et al. proved in [10] that for any irrational rotation number the conjugating map is singular. The rigidity problem for the *break equivalent* (see Definition 7.1) $C^{2+\alpha}$-homeomorphisms with several break points and with trivial total jumps (i.e. the product of jump ratios is equal to 1) was studied by Cunha and Smania in [7]. They have proved that any two such homeomorphisms with bounded combinatorics are C^1-conjugated. The main idea of this work is to consider piecewise-smooth circle homeomorphisms as generalized interval exchange transformations. The case of non break equivalent homeomorphisms with two break points was studied by Akhadkulov et al. in [2,3]. It was shown that if two homeomorphisms