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Irreducibility of Laguerre Polynomial $L_n^{(-1-n-r)}(x)$

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Abstract

We consider the algebraic properties of Generalized Laguerre Polynomials for negative integral values given by $L_n^{(-1-n-r)}(x) = \sum_{j=0}^{n} \binom{n-j+r}{n-j} \frac{x^j}{j!}$. For different values of r, this family gives polynomials which are of great interest. Improving on the earlier results of Hajir and Sell, we prove that $L_n^{(-1-n-r)}$ is irreducible and compute its Galois group for $r \leq 22$. Also we prove that $L_n^{(-1-n-r)}$ is irreducible and its Galois group contains A_n whenever $n > \frac{r}{1.63} e^{1.00008r}$.

Keywords: Galois group; Irreducibility; Laguerre Polynomials; Primes

1. Introduction

For a real number α and an integer $n \geq 1$, the Generalized Laguerre Polynomials (GLP) are a family of polynomials defined by

$$L_n^{(\alpha)}(x) = (-1)^n \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-x)^j}{j!}.$$

The Generalized Laguerre Polynomials have been extensively studied in various branches of analysis and mathematical physics where they play an important role. The algebraic properties

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of GLP were first studied by Schur [10,11] where he established the irreducibility of $L_n^{(\alpha)}(x)$ for $\alpha \in \{0, 1, -n-1\}$, gave a formula for the discriminant $\Delta_n^{(\alpha)}$ of $\mathcal{L}_n^{(\alpha)}(x) = n! L_n^{(\alpha)}(x)$ by

$$\Delta_n^{(\alpha)} = \prod_{j=1}^{n} j^j (\alpha + j)^{j-1}$$

and calculated their associated Galois groups. For an account of results obtained on GLP, we refer to Hajir [6] and Filaseta [1,4].

We shall restrict α to a negative integer in this paper. For integer $r \geq 0$, we consider

$$\begin{aligned} L_n^{\langle r \rangle}(x) &:= L_n^{(-1-n-r)}(x) \\ &= (-1)^n \sum_{j=0}^{n} \binom{-r-1}{n-j} \frac{(-x)^j}{j!} = \sum_{j=0}^{n} \binom{n-j+r}{n-j} \frac{x^j}{j!} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}_n^{\langle r \rangle}(x) &:= n! L_n^{(-1-n-r)}(x) \\ &= \sum_{j=0}^{n} \binom{n}{j} (r+1)(r+2)\cdots(r+n-j) x^j. \end{aligned}$$

By a factor of $L_n^{\langle r \rangle}(x)$, we always mean its factor over $\mathbb{Q}$. We denote

$$\Delta_n^{\langle r \rangle} := \Delta_n^{(-1-n-r)} = \prod_{j=1}^{n} j^j (-1-n-r+j)^{j-1}. \tag{1}$$

We observe that $\Delta_n^{\langle r \rangle}$ is an integer since r is an integer. We denote by $G_n(r)$ the Galois group of $\mathcal{L}_n^{\langle r \rangle}(x)$ over $\mathbb{Q}$. We observe that $G_1(r) = S_1 = A_1$ where S_n denotes the permutation group and A_n the alternating group on n symbols. Thus we shall always write $G_1(r) = S_1$. Schur [10,11] proved that $L_n^{\langle 0 \rangle}(x)$ is irreducible and has Galois group A_n or S_n according as n is divisible by 4 or not, respectively. We observe that $L_n^{\langle 0 \rangle}(x)$ is the truncated exponential series. Coleman [2] gave a different proof for this result. In fact the method given by Coleman [2], further developed and refined by Filaseta [4], turns out to be very powerful for studying irreducibility of GLP. The case $r = n$ gives Bessel polynomials and Filaseta and Trifonov [5] proved their irreducibility for all n. It has been proved by Hajir [6] and Sell [9] that $L_n^{\langle r \rangle}(x)$ is irreducible and its Galois group contains A_n when $r = 1$ or $r = 2$, respectively. More precisely, Hajir [6] proved that for $n \geq 14$,

$$G_n(1) = \begin{cases} A_n & \text{if } n \equiv 0 \pmod 4 \\ S_n & \text{otherwise} \end{cases}$$

and Sell [9] proved that for $n \geq 14$,

$$G_n(2) = \begin{cases} A_n & \text{if } n = 4k(k+1) \text{ for } k \text{ a positive integer} \\ S_n & \text{otherwise.} \end{cases}$$

For $n < 14$, we check that $G_n(1) = G_n(2) = S_n$ unless $(n, r) = \{(5, 1), (9, 1), (13, 1), (8, 2)\}$ in which case $G_n(r) = A_n$. Further Hajir [6] showed that for $3 \leq r \leq 8$ and $n \geq 1$, $L_n^{\langle r \rangle}(x)$ is irreducible and $G_n(r)$ contains A_n. Also he proved that for $n > B(r)$ and $r \geq 9$, $L_n^{\langle r \rangle}(x)$ is