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On the chromatic number of the power graph of a finite group

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Abstract

The power graph Γ_G of a finite group G is the graph whose vertex set is the group, two distinct elements being adjacent if one is a power of the other. We investigate the chromatic number $\chi(\Gamma_G)$ of Γ_G. A characterization of $\chi(\Gamma_G)$ is presented, and a conjecture in Mirzargar et al. (2012) is disproved. Moreover, we classify all finite groups whose power graphs are uniquely colorable, split or unicyclic.

Keywords: Finite group; Power graph; Graph coloring; Split graph; Unicyclic graph

1. Introduction

All groups considered in this paper are finite. Kelarev and Quinn [14] introduced the *directed power graph* of a group G, which is the digraph with vertex set G and there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m. The directed power graphs were also considered in [15,17,16]. Motivated by this, Chakrabarty, Ghosh and Sen [4] introduced the (undirected) power graphs. The *power graph* Γ_G of a group G is the graph whose vertex set is G with two distinct vertices adjacent if one is a power of the other. Recently, many interesting results on the power graphs have been obtained, see [6,2,3,5,8,11,10,20,21,18]. In [1], Abawajy, Kelarev and Chowdhury gave a detailed list of results and open questions.

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For a graph Γ, by $V(\Gamma)$ and $E(\Gamma)$ we denote its vertex set and its edge set, respectively. The *chromatic number* of Γ, denoted by $\chi(\Gamma)$, is the smallest number of colors needed to color the vertices of Γ so that no two adjacent vertices share the same color. Denote by $\mathbb{Z}_n$ the cyclic group of order n. Mirzargar, Ashrafi and Nadjafi-Arani [20] computed the chromatic number of $\mathbb{Z}_n$ and proposed the following conjecture.

Conjecture 1.1 ([20, Conjecture 1]). *Let G be a group. Then*

$$\chi(\Gamma_G) = \chi(\Gamma_{\mathbb{Z}_n}),$$

where n is the maximum order of an element in G.

Motivated by the conjecture, we shall explore the coloring of the power graph of a non-cyclic group.

A partition of the vertex set of a graph Γ is called a *coloring* if each set of the partition is an independent set of Γ. If there is a unique partition of $V(\Gamma)$ into $\chi(\Gamma)$ independent sets, then Γ is said to be *uniquely colorable*. For more information on the uniquely colorable graph, see [13, Chapter 6, p. 113].

The *clique number* $\omega(\Gamma)$ of a graph Γ is the maximum size of a clique in a graph. If $\chi(\Delta) = \omega(\Delta)$ for each induced subgraph Δ of Γ, then Γ is called a *perfect graph*. It was noted in [9, Theorem 1] and [11, Corollary 2.5] that all power graphs are perfect. A graph Γ is said to be *split* if $V(\Gamma)$ is the disjoint union of a clique and an independent set. Split graphs form a very useful class of perfect graphs. More information on the split graphs, can be found in [12,19]. A graph is called *unicyclic* if it is connected and has a unique cycle.

In this paper, we disprove Conjecture 1.1 and characterize the chromatic number of the power graph of a finite group. Furthermore, we classify all finite groups whose power graphs are uniquely colorable, split or unicyclic.

2. Coloring

In the following proposition the chromatic number of the power graph of a cyclic group is determined.

Proposition 2.1 ([20, Theorem 2]). *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1 < p_2 < \cdots < p_r$ are prime numbers. Then*

$$\chi(\Gamma_{\mathbb{Z}_n}) = p_r^{\alpha_r} + \sum_{j=0}^{r-2} (p_{r-j-1}^{\alpha_{r-j-1}} - 1)\Big(\prod_{i=0}^{j} \phi(p_{r-i}^{\alpha_{r-i}})\Big),$$

where ϕ is Euler's totient function.

Denote by $|x|$ the order of element x in the group. Now we disprove Conjecture 1.1 by the following example.

Example 2.2. Given a group G and an element $g \in G$ of order n, $\Gamma_{\langle g\rangle}$ is an induced subgraph of Γ_G which is isomorphic to $\Gamma_{\mathbb{Z}_n}$. It follows that $\chi(\Gamma_G) \geq \chi(\Gamma_{\langle g\rangle}) = \chi(\Gamma_{\mathbb{Z}_n})$. Since power graphs are perfect, Conjecture 1.1 is equivalent to the statement that $\Gamma_{\langle g\rangle}$ contains a maximal clique of Γ_G whenever g has maximum order. In fact, this is not always the case.

Our counterexample to Conjecture 1.1 is the general linear group $\mathrm{GL}(2, p)$ of invertible 2×2 matrices over $\mathrm{GF}(p)$ for certain prime numbers p. By [22, Theorem 2], the maximum order