Available online at www.sciencedirect.com

ScienceDirect

Indagationes Mathematicae 26 (2015) 646–659

indagationes mathematicae

www.elsevier.com/locate/indag

Some operator Bellman type inequalities

Mojtaba Bakherad[a,*], Ali Morassaei[b]

[a] Department of Mathematics, Faculty of Mathematics, University of Sistan and Baluchestan, Zahedan, Iran

[b] Department of Mathematics, Faculty of Sciences, University of Zanjan, University Blvd., Zanjan 45371-38791, Iran

Received 17 February 2015; received in revised form 13 April 2015; accepted 29 April 2015

Communicated by H. Woerdeman

Abstract

In this paper, we employ the Mond–Pečarić method to establish some reverses of the operator Bellman inequality under certain conditions. In particular, we show

$$\delta I_{\mathscr{H}} + \sum_{j=1}^{n} \omega_j \Phi_j \left((I_{\mathscr{H}} - A_j)^p \right) \geq \left(\sum_{j=1}^{n} \omega_j \Phi_j (I_{\mathscr{H}} - A_j) \right)^p ,$$

where A_j $(1 \leq j \leq n)$ are self-adjoint contraction operators with $0 \leq mI_{\mathscr{H}} \leq A_j \leq MI_{\mathscr{H}}$, Φ_j are unital positive linear maps on $\mathbb{B}(\mathscr{H})$, $\omega_j \in \mathbb{R}_+$ $(1 \leq j \leq n)$ such that $\sum_{j=1}^{n} \omega_j = 1$, $\delta = (1-p)\left(\frac{1}{p}\frac{(1-m)^p-(1-M)^p}{M-m}\right)^{\frac{p}{p-1}} + \frac{(1-M)(1-m)^p-(1-m)(1-M)^p}{M-m}$ and $0 < p < 1$. We also present some refinements of the operator Bellman inequality.

Keywords: Bellman inequality; Operator mean; The Mond–Pečarić method; Positive linear map

1. Introduction

Let $\mathbb{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with the identity $I_{\mathscr{H}}$. In the case when $\dim \mathscr{H} = n$, we identify $\mathbb{B}(\mathscr{H})$ with the matrix

* Corresponding author.

E-mail addresses: mojtaba.bakherad@yahoo.com, bakherad@member.ams.org (M. Bakherad), morassaei@znu.ac.ir, morassaei@chmail.ir (A. Morassaei).

http://dx.doi.org/10.1016/j.indag.2015.04.006

algebra $\mathcal{M}_n(\mathbb{C})$ of all $n \times n$ matrices with entries in the complex field. An operator $A \in \mathbb{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$ and in this case we write $A \geq 0$. We write $A > 0$ if A is a positive invertible operator. The set of all positive invertible operators is denoted by $\mathbb{B}(\mathscr{H})_+$. For self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$, we say $A \leq B$ if $B - A \geq 0$. Also, an operator $A \in \mathbb{B}(\mathscr{H})$ is said to be contraction, if $A^*A \leq I_{\mathscr{H}}$. The Gelfand map $f(t) \mapsto f(A)$ is an isometrical $*$-isomorphism between the C^*-algebra $C(\mathrm{sp}(A))$ of continuous functions on the spectrum $\mathrm{sp}(A)$ of a self-adjoint operator A and the C^*-algebra generated by A and $I_{\mathscr{H}}$. If $f, g \in C(\mathrm{sp}(A))$, then $f(t) \geq g(t)$ $(t \in \mathrm{sp}(A))$ implies that $f(A) \geq g(A)$.

Let f be a continuous real valued function defined on an interval J. It is called operator monotone if $A \leq B$ implies $f(A) \leq f(B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J; see [4] and references therein for some recent results. It is said to be operator concave if $\lambda f(A)+(1-\lambda)f(B) \leq f(\lambda A+(1-\lambda)B)$ for all self-adjoint operators $A, B \in \mathbb{B}(\mathscr{H})$ with spectra in J and all $\lambda \in [0, 1]$. Every nonnegative continuous function f is operator monotone on $[0, +\infty)$ if and only if f is operator concave on $[0, +\infty)$; see [5, Theorem 8.1]. A map $\Phi : \mathbb{B}(\mathscr{H}) \longrightarrow \mathbb{B}(\mathscr{K})$ is called positive if $\Phi(A) \geq 0$ whenever $A \geq 0$, where $\mathscr{K}$ is a complex Hilbert space and is said to be unital if $\Phi(I_{\mathscr{H}}) = I_{\mathscr{K}}$. We denote by $\mathbf{P}_N[\mathbb{B}(\mathscr{H}), \mathbb{B}(\mathscr{K})]$ the set of all unital positive linear maps $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{K})$.

The axiomatic theory for operator means of positive invertible operators have been developed by Kubo and Ando [7]. A binary operation σ on $\mathbb{B}(\mathscr{H})_+$ is called a connection, if the following conditions are satisfied:

(i) $A \leq C$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$;
(ii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$, where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;
(iii) $T^*(A\sigma B)T \leq (T^*AT)\sigma(T^*BT)$ $(T \in \mathbb{B}(\mathscr{H}))$.

There exists an affine order isomorphism between the class of connections and the class of positive operator monotone functions f defined on $(0, \infty)$ via $f(t)I_{\mathscr{H}} = I_{\mathscr{H}}\sigma_f(tI_{\mathscr{H}})$ $(t > 0)$. In addition, $A\sigma_f B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ for all $A, B \in \mathbb{B}(\mathscr{H})_+$. The operator monotone function f is called the representing function of σ_f. A connection σ_f is a mean if it is normalized, i.e. $I_{\mathscr{H}}\sigma_f I_{\mathscr{H}} = I_{\mathscr{H}}$. The function $f_{\nabla_\mu}(t) = (1-\mu)+\mu t$ and $f_{\sharp_\mu}(t) = t^\mu$ on $(0, \infty)$ for $\mu \in (0, 1)$ give the operator weighted arithmetic mean $A\nabla_\mu B = (1-\mu)A+\mu B$ and the operator weighted geometric mean $A\sharp_\mu B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^\mu A^{\frac{1}{2}}$, respectively. The case $\mu = 1/2$, the operator weighted geometric mean gives rise to the so-called geometric mean $A\sharp B$.

Bellman [2] proved that if p is a positive integer and a, b, a_j, b_j $(1 \leq j \leq n)$ are positive real numbers such that $\sum_{j=1}^n a_j^p \leq a^p$ and $\sum_{j=1}^n b_j^p \leq b^p$, then

$$\left(a^p - \sum_{j=1}^n a_j^p\right)^{1/p} + \left(b^p - \sum_{j=1}^n b_j^p\right)^{1/p} \leq \left((a+b)^p - \sum_{j=1}^n (a_j+b_j)^p\right)^{1/p}.$$

A multiplicative analogue of this inequality is due to J. Aczél; see [1] and its operator version in [10]. In 1956, Aczél [1] proved that if a_j, b_j $(1 \leq j \leq n)$ are positive real numbers such that $a_1^2 - \sum_{j=2}^n a_j^2 > 0$ or $b_1^2 - \sum_{j=2}^n b_j^2 > 0$, then

$$\left(a_1^2 - \sum_{j=2}^n a_j^2\right)\left(b_1^2 - \sum_{j=2}^n b_j^2\right) \leq \left(a_1 b_1 - \sum_{j=2}^n a_j b_j\right)^2.$$