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Real quadratic double sums

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Abstract

In 1988, Andrews, Dyson and Hickerson initiated the study of q-hypergeometric series whose coefficients are dictated by the arithmetic in real quadratic fields. In this paper, we provide a dozen q-hypergeometric double sums which are generating functions for the number of ideals of a given norm in rings of integers of real quadratic fields and prove some related identities.

Keywords: Bailey pairs; Real quadratic fields; q-series

1. Introduction

In 1988, Andrews, Dyson and Hickerson [2] initiated the study of q-hypergeometric series whose coefficients are dictated by the arithmetic in real quadratic fields. They considered a q-series from Ramanujan's lost notebook,

$$\sigma(q) := \sum_{n\geq 0} \frac{q^{\binom{n+1}{2}}}{(-q)_n}, \tag{1.1}$$

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and proved the Hecke-type identity,

$$\sigma(q) = \sum_{\substack{n\geq 0 \\ -n\leq j\leq n}} (-1)^{n+j} q^{n(3n+1)/2-j^2}(1-q^{2n+1}). \tag{1.2}$$

Here and throughout we assume that $|q| < 1$ and use the standard q-hypergeometric notation,

$$(a)_n = (a;q)_n = \prod_{k=1}^{n}(1-aq^{k-1}),$$

valid for $n \in \mathbb{N}\cup\{\infty\}$. They then used identity (1.2) to relate the coefficients of $\sigma(q)$ to the ring of integers of the real quadratic field $\mathbb{Q}(\sqrt{6})$. As a consequence, they found that these coefficients satisfy an "almost" exact formula, are lacunary and yet, surprisingly, assume all integer values infinitely often.

Other rare and intriguing examples of q-series related to real quadratic fields (predicted to exist by Dyson [6]) have been discovered over the years (see [3,5,7,8], for example). The key in each of these cases is the use of Bailey pairs to prove a Hecke-type identity resembling (1.2). We recall that a *Bailey pair* relative to a is a pair of sequences $(\alpha_n, \beta_n)_{n\geq 0}$ satisfying

$$\beta_n = \sum_{k=0}^{n} \frac{\alpha_k}{(q)_{n-k}(aq)_{n+k}}. \tag{1.3}$$

For example, Bringmann and Kane [3] discovered the following two Bailey pairs. First, (a_n, b_n) is a Bailey pair relative to 1, where

$$a_{2n} = (1-q^{4n})q^{2n^2-2n}\sum_{j=-n}^{n-1} q^{-2j^2-2j}, \tag{1.4}$$

$$a_{2n+1} = -(1-q^{4n+2})q^{2n^2}\sum_{j=-n}^{n} q^{-2j^2}, \tag{1.5}$$

and

$$b_n = \frac{(-1)^n (q;q^2)_{n-1}}{(q)_{2n-1}}\chi(n\neq 0). \tag{1.6}$$

Second, (α_n, β_n) is a Bailey pair relative to q, where

$$\alpha_{2n} = \frac{1}{1-q}\left(q^{2n^2+2n}\sum_{j=-n}^{n-1} q^{-2j^2-2j} + q^{2n^2}\sum_{j=-n}^{n} q^{-2j^2}\right), \tag{1.7}$$

$$\alpha_{2n+1} = -\frac{1}{1-q}\left(q^{2n^2+4n+2}\sum_{j=-n}^{n} q^{-2j^2} + q^{2n^2+2n}\sum_{j=-n-1}^{n} q^{-2j^2-2j}\right), \tag{1.8}$$

and

$$\beta_n = \frac{(-1)^n (q;q^2)_n}{(q)_{2n+1}}. \tag{1.9}$$

Recently, we showed that (1.4)–(1.9) are actually special cases of a much more general result (see Theorems 1.1–1.3 in [9]). This led to new Bailey pairs involving indefinite quadratic forms,