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On some new integral inequalities of Gronwall–Bellman–Bihari type with delay for discontinuous functions and their applications[☆]

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Abstract

In this article, some new explicit bounds on solutions to a class of new nonlinear integral inequalities of Gronwall–Bellman–Bihari type with delay for discontinuous functions are established. These inequalities generalize and improve some former famous results about inequalities, and which provide an excellent tool to discuss the qualitative and quantitative properties for solutions to some nonlinear differential and integral equations. To illustrate our results, we present an example to show estimated solutions for an impulsive differential system.

Keywords: Integral inequalities; Discontinuous functions; Gronwall–Bellman–Bihari type inequalities; Impulsive differential systems

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1. Introduction

During the past years, a number of efforts have been devoted to the integral inequalities. The celebrated results given by Gronwall, Bellman, Bihari and their linear and nonlinear generalizations in the case of continuous and discontinuous functions provide a fundamental role in the study of many qualitative properties of differential and integral equations, which we can find in [3,6,4,11,12,15–17,2,1,10,9]. For some new development on this topic, see [5,8,13,18,14,19,7].

In 2007, Jiang and Meng [8] discussed the following delay integral inequalities:

$$x^p(t) \leq \rho(t) + \pi(t) \int_{t_0}^{t} \Big[f(s) x^q(s) + h(s) x^r(\sigma(s)) \Big] ds$$

with the initial condition:

$$\begin{aligned} & s(t) = \phi(t) \quad t \in R_+ \\ & \phi(\sigma(t)) \leq \rho(t)^{\frac{1}{p}} \quad t \in R_+ \ \sigma(t) \leq 0 \end{aligned}$$

and

$$x^p(t) \leq \rho(t) + \pi(t) \int_{t_0}^{t} \Big[f(s) x^q(s) + L\Big(s, x(\sigma(s))\Big) \Big]$$

with the same initial condition. Under the assumption of functions $x(t), \rho(t), \pi(t), f(t), h(t) \in C(R^+, R^+)$, authors obtained explicit bounds on unknown function $x(t)$ of the above inequalities.

The purpose of this paper is to give explicit bounds to some new nonlinear integral inequalities of Gronwall–Bellman–Bihari type with delay for discontinuous functions. Precisely, we consider the following nonlinear integral inequalities:

$$x^p(t) \leq \rho(t) + \pi(t) \int_{t_0}^{t} \Big[f(s) x^q(s) + h(s) x^r(\sigma(s)) \Big] ds + \sum_{to < t_i < t} a_i x^m(t_i - 0)$$

and

$$x^p(t) \leq \rho(t) + \pi(t) \int_{t_0}^{t} \Big[f(s) x^q(s) + L\Big(s, x(\sigma(s))\Big) \Big] + \sum_{to < t_i < t} a_i x^m(t_i - 0).$$

Our results develop and improve certain results which were proved by Jiang and Meng [8]. At the end of the article, an example of application is presented to show estimated solutions for an impulsive differential system.

2. Main results

Lemma 2.1 *([8]). Assume that* $a \geq 0,\ \ p \geq q \geq 0$ *and* $p \neq 0$*, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}}$$

for any $K > 0$.