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Some reverse inequalities for matrices on indefinite inner product spaces

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Abstract

In this paper, we generalize the following inequality due to N. Bebiano et al which states that for J-contraction $T \in \mathcal{M}_n$

$$|[Tx, y]_J|^2 \geq \left[|T|_J^{2\alpha} x, x\right]_J \left[|T^\sharp|_J^{2(1-\alpha)} y, y\right]_J$$

for any $x, y \in C^n$ with x or y time-like and $\alpha \in [0, 1]$. The above inequality gives a reverse weighted mixed Schwarz inequality. Also, we prove some reverse inequalities for indefinite inner product space. Moreover, we establish some refinements for the reverse of Schwarz inequality as well.

Keywords: Reverse Schwarz inequality; Reverse Heinz–Kato–Furuta inequality; Indefinite inner product; Minkowski inner product

1. Introduction and preliminaries

The concept of Minkowski space, raised by H. Minkowski and used by in theoretical physics and differential geometry, is based upon the concept of indefinite inner product.

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The *indefinite inner product* on a complex space V is a complex function $[x, y] : V \times V \to \mathbb{C}$ with the following properties:

- $[x + y, z] = [x, z] + [y, z]$,
- $[\lambda x, y] = \lambda[x, y]$ for every $\lambda \in \mathbb{C}$,
- $[x, y] = \overline{[y, x]}$ for $x, y \in V$.
- If $[x, y] = 0$ for every $y \in V$ then $x = 0$.

A vector space V with an indefinite inner product is an indefinite inner product space, see [4,5].

Let us consider the n-dimensional vector space C^n with an indefinite inner product structure induced by the Minkowski inner product

$$[x, y]_J = y^* J x, \quad x, y \in C^n, \tag{1}$$

where $J = diag(-1, 1, \ldots, 1)$ is the metric matrix. We say that x, y are J-*orthogonal* vectors if $[x, y]_J = 0$. A vector $x \in C^n$ is said to be time-like, space-like and light-like (or isotropic) if $[x, x]_J < 0$, $[x, x]_J > 0$ and $[x, x]_J = 0$, respectively [2].

We denote the algebra of $n \times n$ complex matrices by $\mathcal{M}_n$. Let J be a Hermitian involution i.e., $J = J^* = J^{-1}$. The J-*adjoint* of $A \in \mathcal{M}_n$ is the unique matrix $A^\sharp$ satisfying $[Ax, y]_J = [x, A^\sharp y]_J$ for $x, y \in C^n$ which is equivalent to say that $A^\sharp = JA^*J$, since J is a Hermitian involution. A matrix $A \in \mathcal{M}_n$ is said to be J-*Hermitian* if $A^\sharp = A$ and an invertible matrix $U \in \mathcal{M}_n$ such that $U^{-1} = U^\sharp$ is said to be J-*unitary*. For a pair J-Hermitian matrices $A, B \in \mathcal{M}_n$, the J-*order* relation, denoted by $A \geq^J B$, is defined by

$$[Ax, x]_J \geq [Bx, x]_J, \quad (x \in C^n).$$

It is obvious that $A \geq^J B$ if and only if $JA \geq JB$.

As it was mentioned in [1] that all eigenvalues of a J-Hermitian matrix $A \in \mathcal{M}_n$ are not always real. Moreover, the eigenvalues of a J-Hermitian matrix $A \in \mathcal{M}_n$ such that $I_n \geq^J A$ are all real. Also, a J-Hermitian matrix $A \in \mathcal{M}_n$ is J-*positive* if $A \geq^J 0$.

If $I_n \geq^J A$ and all the eigenvalues of A are non-negative, then the (J-Hermitian) α power of A for $0 < \alpha < 1$ is given by the integral

$$A^\alpha = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \zeta^{\alpha-1} A(\zeta I_n + A)^{-1} d\zeta$$

(see [7, Lemma 2.1]).

A matrix $A \in \mathcal{M}_n$ is called J-*contraction* if $[x, x]_J \geq [Ax, Ax]_J$ for any $x \in C^n$. Note that when A is J-contraction then all the eigenvalues of $A^\sharp A$ are non-negative. Therefore, the J-Hermitian square root of $T^\sharp T$ is well defined and T admits a J-*polar decomposition* $T = U|T|_J$ where U is a J-unitary matrix and $|T|_J = (T^\sharp T)^{\frac{1}{2}}$ is called J-*modulus* of T (see [1]).

In a recent paper N. Bebiano et al. [2] established the following inequality for J-contraction $T \in \mathcal{M}_n$

$$|[Tx, y]_J|^2 \geq \left[|T|_J^{2\alpha} x, x\right]_J \left[|T^\sharp|_J^{2(1-\alpha)} y, y\right]_J \tag{2}$$

for any $x, y \in C^n$ with x or y time-like and $\alpha \in [0, 1]$. The above inequality is a reverse weighted mixed Schwarz inequality which gives a generalization for the following reverse Schwarz inequality

$$|[x, y]_J|^2 \geq [x, x]_J [y, y]_J \tag{3}$$