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The countable lifting property for Riesz space surjections

Anthony W. Hager[a,*], Robert Raphael[b]

[a] *Department of Math & CS, Wesleyan University, Middletown, CT 06457, USA*
[b] *Department of Math & Stat., Concordia University, Montreal, Que H3G 2T6, Canada*

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Abstract

A surjective homomorphism $A \overset{\varphi}{\twoheadrightarrow} B$ of Riesz spaces (real vector lattices) has the "countable lifting property" CLP if: For each countable pairwise disjoint $\{b_n\}$ in B, there are disjoint $\{a_n\}$ in A with $\varphi(a_n) = b_n$ for each n. Previous thoughts on this are due to Topping (1965), Conrad (1968), and in considerable depth, Moore (1970), (and little subsequent, to our knowledge). Here, we consider the issue mostly (not entirely) for Riesz spaces resembling $C(X)$'s. We show (inter alia): $A \overset{\varphi}{\twoheadrightarrow} B$ will have CLP if (a) B is laterally σ-complete; or if (b) $B = C(Y)$ for Y locally compact and σ-compact; or if (c) A is an f-algebra with identity, which is archimedean and uniformly complete, and B is (merely) archimedean (e.g., $A = C(X)$ and $B = C(Y)$, for any X, Y). The main technical device is the notion: b is a weak supremum of $\{b_n\}$ if $b = \bigvee \lambda_n b_n$ for some $\{\lambda_n\} \subseteq (0, +\infty)$.

Keywords: Riesz space; Lifting disjoint sets; Archimedean; Weakly laterally σ-complete; Disjoint σ-property

0. Introduction

Our main reference for Riesz Spaces is [17] which see for terms undefined here.

For a surjective homomorphism (of Riesz spaces) $A \overset{\varphi}{\twoheadrightarrow} B$, the CLP is defined in our Abstract.

* Corresponding author.
E-mail addresses: ahager@wesleyan.edu (A.W. Hager), raphael@alcor.concordia.ca (R. Raphael).

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Topping [22] asserted that the CLP always holds. Conrad [3] gave counterexamples, including one with A archimedean.

Moore [20] mounted a serious attack on the issue, showing (among other things): A is relatively uniformly complete with every $A \overset{\varphi}{\twoheadrightarrow} B$ having CLP iff A is the (real) functions of compact support on a discrete space; there are $A \overset{\varphi}{\twoheadrightarrow} B$ failing CLP with A and B archimedean, even with A super Dedekind complete. He largely (not totally) focused on properties of A yielding CLP. Here, we shall largely (not totally) focus on properties of B.

Our most important definitions and notations will occur first at various places in the paper, but we collect them here for convenient reference.

1. Definitions, notations, etc.

All objects and homomorphisms are in Riesz spaces.

(1) Countable families $\{b_n \mid n = 1, 2, \ldots\} \subseteq B$ will be abbreviated $\{b_n\}$. If $\{b_n\} \subseteq B^+$ is pairwise disjoint ($b_m \wedge b_n = 0$ if $m \neq n$), we say $\{b_n\}$ is cpd (countable, pairwise disjoint).

(2) The CLP for $A \twoheadrightarrow B$ is defined in the Abstract. B has the *Range CLP* (RCLP) if every $A \twoheadrightarrow B$ has CLP.

(3) "Relatively uniformly complete" is abbreviated "r.u. complete".

(4) B is laterally σ-complete, abbreviated $L(\sigma)$ if each cpd $\{b_n\}$ has the supremum in B.

(5) $\{\lambda_n\}$ always denotes a countable indexed subset of $(0, +\infty)$ (in the reals $\mathbb{R}$).

Suppose $\{b_n\} \subseteq B^+$. If there is $\{\lambda_n\}$ (resp., $\{\lambda_n\}$ and $b \in B$) for which $\bigvee \lambda_n b_n$ exists (resp., $\lambda_n b_n \leq b$ for all n), we say that $\{b_n\}$ has a *weak supremum* (resp., b is a *weak bound* for $\{\lambda_n\}$).

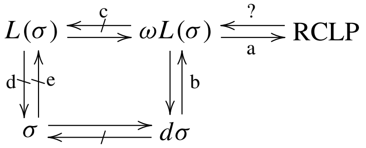
(6) B has the σ-property "σ" (respectively, disjoint σ-property "$d\sigma$"; respectively, is weakly laterally σ-complete "$\omega L(\sigma)$") if each $\{b_n\} \subseteq B^+$ has a weak bound (resp., each cpd $\{b_n\}$ has a weak bound; respectively, each cpd $\{b_n\}$ has a weak supremum).

(7) A weak (resp., strong) unit in B is a $u \in B^+$ for which $|b| \wedge u = 0$ implies $b = 0$ (resp., $I(u) = B$). Here, $I(u) = \{b \in B \mid \exists n(|b| \leq nu)\}$.

The definitions of weak sup, $\omega L(\sigma)$, and $d\sigma$ are new. Everything else above except $L(\sigma)$ can be found in [17]. For systems in $\mathbb{R}^{\mathbb{R}}$, the property σ (also, r.u. convergence) seems to have been introduced in [19] (the "10" is "1910"). Other remarks about the origins of σ appear in [17, (p. 479)] and [7]. $L(\sigma)$ is discussed (not for the first time) in [1,11]. See also [5].

Various other definitions such as RCLP are possible, e.g., Domain CLP or DCLP, or "A has DCLP for archimedean B", or "A has DCLP for complete φ", etc. While not using this terminology, we shall encounter some of these notions in passing.

The following chart sums up some relationships. In it, "$\rightarrow$" means "implies" (or, one class is included in another), "$\not\rightarrow$" means "does not imply", and "$\overset{m}{\rightarrow}$" refers to notes below, where we locate the proof or example in this paper. (Unlabeled $\rightarrow$ are obvious.) "$\overset{?}{\leftarrow}$" means we do not know.



a. This is a main theorem, 4.1(b) below.