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A new fixed point theorem in the fractal space

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Abstract

In this paper we present some important generalizations of the Banach contraction principle in which the Lipschitz constant k is replaced by some real-valued control function.

For the applications to the fractal, we obtain the fixed point theorem of the some generalized contraction in the fractal space.

Keywords: Fixed points; Generalized contraction; Hausdorff distance; Fractal; Topological equivalent

1. Introduction and preparatory results

The most well known result in the theory of fixed points is Banach's contraction mapping principle.

Boyd and Wong [6] and Matkowski (see [11,16,19,20]) proved the theorem of existence of fixed point of mappings in complete metric space. In particular, these theorems extend the result of F.E. Browder's fixed point theorem.

They discussed the Banach contraction principle with some generalized contraction conditions and weakened the usual contraction condition. The main idea in the generalization of Banach's contraction theorem was to use the combining of the ideas in the contraction principle (see [1–5,7–10,12,14–22]).

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If all f_i $(i = 1, \ldots, N)$ are Banach contractions, then the mapping

$$F(A) := \bigcup_{i=1}^{N} f_i(A) \quad \text{for } A \in H(X)$$

has a unique fixed point K in $H(X)$. Here $H(X)$ denotes the space whose points are the compact subsets of the complete metric space (X, d) other than the empty set. This result was proposed by Barnsley [1,2]. The previous works proved theorem dealing with finite (or infinite) families of Matkowski's contractions [16,19,20], which extended the result of Barnsley.

The aim of this paper is to obtain the fixed point theorems of the some generalized contractions and present the application of fixed point theorems in fractal space.

Before we establish the fixed point theorems of the some generalized contraction, we discuss some basic results.

Theorem 1.1 (*Boyd and Wong's Fixed Point Theorem (See [6])*)**.** *Let X be a complete metric space, and let $f : X \to X$ satisfy*

$$d(f(x), f(y)) \le \varphi(d(x, y)) \quad \textit{for all } x, y \in X,$$

where $P = \{d(x, y)|x, y \in X\}$, $\overline{P}$ is the closure P, $\varphi : \overline{P} \to [0, +\infty)$ is upper semicontinuous function from the right on $\overline{P}$ (not necessarily non-decreasing), and satisfies $\varphi(t) < t$ for all $t \in \overline{P}/\{0\}$. Then, f has a unique fixed point $p \in X$, and $\lim_{n\to+\infty} f^n(x) = p$ for each $x \in X$.

J.R. Jachymski [11] obtained a complete characterization of relations between fixed point theorems.

Definition 1.1 (*[11]*)**.** Given a function $\varphi : R_+ \to R_+$ such that $\varphi(t) < t$ for $t > 0$, and a selfmap f of a metric space (X, d), we say that f is φ-contractive if

$$d(f(x), f(y)) \le \varphi(d(x, y)) \quad \text{for all } x, y \in X.$$

Theorem 1.2 (*cf. [11]*)**.** *Let f be a self-map of a metric space (X, d). The following statements are equivalent.*

(1) *There exists an increasing and right continuous function $\varphi : R_+ \to R_+$ such that f is φ-contractive.*

(2) *There exists a continuous function $\psi : R_+ \to R_+$ with $\psi(t) > 0$ for $t > 0$, such that*

$$d(f(x), f(y)) \le d(x, y) - \psi(d(x, y)) \quad \textit{for all } x, y \in X.$$

(3) *There exists an upper semicontinuous function $\varphi : R_+ \to R_+$ such that f is φ-contractive.*

(4) *There exists a function $\varphi : R_+ \to R_+$ with $\limsup_{s\to t}\varphi(s) < t$, for all $t > 0$, such that f is φ-contractive.*

(5) *There exists a strictly increasing function $\varphi : R_+ \to R_+$ such that $\lim_{n\to+\infty} \varphi^n(t) = 0$ for all $t \in R_+$ holds and f is φ-contractive.*

(6) *There exists a strictly increasing and continuous function $\varphi : R_+ \to R_+$ such that f is φ-contractive.*

Remark 1.1 (*See [11]*)**.** Theorem 1.2(3) is the specialization of the condition due to Boyd and Wong [6] in which, originally, φ is right upper semi-continuous.