Available online at www.sciencedirect.com

ScienceDirect

Indagationes Mathematicae 27 (2016) 94–99

indagationes mathematicae

www.elsevier.com/locate/indag

A note on weighted homogeneous Siciak–Zaharyuta extremal functions

Barbara Drinovec Drnovšek[a,b,*], Ragnar Sigurdsson[c]

[a] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia
[b] Institute of Mathematics, Physics and Mechanics, Jadranska 19, SI-1000 Ljubljana, Slovenia
[c] Department of Mathematics, School of Engineering and Natural Sciences, University of Iceland, IS-107 Reykjavík, Iceland

Received 4 June 2015; received in revised form 10 July 2015; accepted 3 August 2015

Communicated by J.J.O.O. Wiegerinck

Abstract

We prove that for any given upper semicontinuous function φ on a subset E of $\mathbb{C}^n \setminus \{0\}$, such that the complex cone generated by E minus the origin is open and connected, the homogeneous Siciak–Zaharyuta function with the weight φ on E can be represented as an envelope of a disc functional.

Keywords: Siciak's homogeneous extremal function; Disc functional; Envelope of a disc functional

1. Introduction

Let $\mathcal{L}$ denote the Lelong class on $\mathbb{C}^n$ and $\mathcal{L}^h$ the subclass of functions u which are *logarithmically homogeneous*. Let $\varphi\colon E \to \overline{\mathbb{R}}$ be a function on a subset E of $\mathbb{C}^n$ taking values in the extended real line $\overline{\mathbb{R}}$. The *Siciak–Zaharyuta extremal function* $V_{E,\varphi}$ *with weight* φ is defined by

$$V_{E,\varphi} = \sup\{u \in \mathcal{L};\ u|E \le \varphi\}.$$

* Corresponding author at: Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1000 Ljubljana, Slovenia.

E-mail addresses: barbara.drinovec@fmf.uni-lj.si (B. Drinovec Drnovšek), ragnar@hi.is (R. Sigurdsson).

http://dx.doi.org/10.1016/j.indag.2015.08.001

The *homogeneous Siciak–Zaharyuta extremal function* $V^h_{E,\varphi}$ *with weight* φ is defined similarly with $\mathcal{L}^h$ in the role of $\mathcal{L}$. In the special case when $\varphi = 0$ we only write V_E (and V^h_E) and we call this function the (*homogeneous*) *Siciak–Zaharyuta extremal function for the set* E. The function V_E (V^h_E) is also called the (*homogeneous*) *pluricomplex Green function for* E *with pole at infinity*. Let $\mathbb{C}E = \{\lambda z;\ \lambda \in \mathbb{C}, z \in E\}$ and $\mathbb{C}^*E = \{\lambda z;\ \lambda \in \mathbb{C}^*, z \in E\}$.

Theorem 1. *Let* $\varphi: E \to \mathbb{R} \cup \{-\infty\}$ *be an upper semicontinuous function on a subset* E *of* $\mathbb{C}^n \setminus \{0\}$ *such that* $\mathbb{C}^*E$ *is nonpluripolar. Assume that there exists a function in* $\mathcal{L}^h$ *dominated by* φ *on* E*. Then the largest logarithmically homogeneous function* $\mathbb{C}E \to \mathbb{R} \cup \{-\infty\}$ *dominated by* φ *on* E *is upper semicontinuous on* $\mathbb{C}^*E$ *and it is of the form* $\log \varrho_{E,\varphi}$, *where*

$$\varrho_{E,\varphi}(z) = \inf\{|\lambda| e^{\varphi(z/\lambda)};\ \lambda \in \mathbb{C}^*, z/\lambda \in E\}, \quad z \in \mathbb{C}^*E. \tag{1}$$

If $\mathbb{C}^*E$ *is open and connected, then for every* $z \in \mathbb{C}^n$

$$V^h_{E,\varphi}(z) = \inf\left\{\int_{\mathbb{T}} \log \varrho_{E,\varphi}(f_1, \dots, f_n)\, d\sigma;\ f \in \mathcal{O}(\overline{\mathbb{D}}, \mathbb{P}^n),\ f = [f_0 : \cdots : f_n]\,,\right.$$
$$\left. f(\mathbb{T}) \subset \mathbb{C}^*E,\ f_0(0) = 1,\ (f_1(0), \dots, f_n(0)) = z\right\}. \tag{2}$$

If $\mathbb{C}E = \mathbb{C}^n$, *then for every* $z \in \mathbb{C}^n$

$$V^h_{E,\varphi}(z) = \inf\left\{\int_{\mathbb{T}} \log \varrho_{E,\varphi} \circ f\, d\sigma;\ f \in \mathcal{O}(\overline{\mathbb{D}}, \mathbb{C}^n),\ f(0) = z\right\}. \tag{3}$$

A *closed analytic disc* in a complex space X is a holomorphic map $f: \overline{\mathbb{D}} \to X$ from some neighbourhood of the unit disc $\mathbb{D}$ in $\mathbb{C}$ into X. The point $z = f(0) \in X$ is called the *center* of f. We denote the set of all closed analytic discs in X by $\mathcal{O}(\overline{\mathbb{D}}, X)$. For a subset $\mathcal{B}$ of $\mathcal{O}(\overline{\mathbb{D}}, X)$, let $\mathcal{B}(z)$ consist of all $f \in \mathcal{B}$ with center z. A *disc functional* H on X is a map defined on some subset $\mathcal{A}$ of $\mathcal{O}(\overline{\mathbb{D}}, X)$ with values in the extended real line $\overline{\mathbb{R}}$. The *envelope* $E_{\mathcal{B}}H: X \to \overline{\mathbb{R}}$ of H with respect to the subset $\mathcal{B}$ of $\mathcal{A}$ is defined by $E_{\mathcal{B}}H(z) = \inf\{H(f);\ f \in \mathcal{B}(z)\}$ for $z \in X$.

The formula (2) is an example of a disc envelope formula, where $\mathcal{A}$ consists of all closed analytic discs with values in the projective space, i.e., elements f in $\mathcal{O}(\overline{\mathbb{D}}, \mathbb{P}^n)$, which map the unit circle $\mathbb{T}$ into $\mathbb{C}^*E$, $H(f)$ is the integral, and $\mathcal{B}$ is the subset of $\mathcal{A}$ consisting of discs with $f_0(0) = 1$. We identify a point $[1 : z] \in \mathbb{P}^n$ with the point $z \in \mathbb{C}^n$.

For general information on the Siciak–Zaharyuta extremal function see Siciak [10–14] and Zaharyuta [15]. The first disc envelope formula for V_E was proved by Lempert in the case when E is an open convex subset of $\mathbb{C}^n$ with real analytic boundary. (The proof is given in Momm [7, Appendix].) Lárusson and Sigurdsson [3] proved disc envelope formulas for V_E for open connected subsets E of $\mathbb{C}^n$. Magnússon and Sigurdsson [6] generalized this result and obtained a disc formula for $V_{E,\varphi}$ in the case when φ is an upper semicontinuous function on an open connected subset E of $\mathbb{C}^n$. Drinovec Drnovšek and Forstnerič [1] proved disc envelope formulas for V_E for open subsets E of an irreducible and locally irreducible algebraic subvariety of $\mathbb{C}^n$. Magnússon [5] established disc envelope formulas for the global extremal function in the projective space.

Notation. Let $\mathbb{D}$ denote the open unit disc in $\mathbb{C}$, $\mathbb{T}$ the unit circle, and σ the arc length measure on $\mathbb{T}$ normalized to 1. For a subset X of $\mathbb{C}^n$ we let $\mathcal{USC}(X)$ denote the set of all upper semicontinuous functions on X, and for open subset U of $\mathbb{C}^n$ we denote by $\mathcal{PSH}(U)$ the set of all plurisubharmonic functions on U. Let $\log^+$ denote the positive part of the log function.