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On the growth of solutions of difference equations in ultrametric fields

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Abstract

Let $\mathbb{K}$ be a complete ultrametric algebraically closed field. In this article, we consider the functional equations $\sum_{i=0}^{s} g_i(x)y(q^i x) = h(x)$ and $\sum_{i=0}^{s} g_i(x)y(x+i) = h(x)$, where q is an element of $\mathbb{K}$ such that $0 < |q| < 1$ and $h(x)$, $g_0(x), \ldots, g_s(x)$ $(s \geq 1)$ are meromorphic functions in all $\mathbb{K}$ such that $g_0(x)g_s(x) \not\equiv 0$. For each of the above equations, we study the growth of meromorphic solutions $y = f(x)$ according to that of the functions $g_0, \ldots, g_s$ and h.

Keywords: Ultrametric meromorphic functions; Nevanlinna theory; Difference equations; Order of growth

1. Introduction and preliminary results

Let $\mathbb{K}$ be an algebraically closed field, complete for an ultrametric absolute value. We denote by $\mathcal{A}(\mathbb{K})$ the $\mathbb{K}$-algebra of entire functions in $\mathbb{K}$ and by $\mathcal{M}(\mathbb{K})$ the field of meromorphic functions in $\mathbb{K}$, i.e. the field of fractions of $\mathcal{A}(\mathbb{K})$.

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This paper is devoted to the study of the growth of meromorphic solutions $y = f(x)$ of the functional equation:

$$(E)\ \sum_{i=0}^{s} g_i(x) y(q^i x) = h(x),$$

where $q \in \mathbb{K}$ is such that $0 < |q| < 1$ and $h(x),\ g_0(x), \ldots, g_s(x)$ $(s \geq 1)$ are meromorphic functions such that $g_0(x) g_s(x) \not\equiv 0$.

Indeed, recently many papers (see for instance [1–4,9,10,13]) focused on such equations and many meaningful results have been obtained about the growth of their solutions. The main purpose of this paper is to extend some of these results. We also make a similar study of difference equations of the type:

$$\sum_{i=0}^{s} g_i(x) y(x+i) = h(x).$$

Throughout this paper, we use the ultrametric Nevanlinna theory (see e.g., [7,6]). So we first have to recall some basic notions of this theory.

Given $R > 0$, we denote by $D^-(0, R)$ the open disk $\{x \in \mathbb{K} : |x| < R\}$ and by $D(0, R)$ the closed disk $\{x \in \mathbb{K} : |x| \leq R\}$. Similarly, we denote by $\mathcal{A}(D^-(0, R))$ the $\mathbb{K}$-algebra of analytic functions inside the disk $D^-(0, R)$, i.e. the set of power series converging inside $D^-(0, R)$ and by $\mathcal{M}(D^-(0, R))$ the field of meromorphic functions in $D^-(0, R)$, i.e. the field of fractions of $\mathcal{A}(D^-(0, R))$.

For every $r \in\ ]0, R[$, we define a multiplicative norm $|\,|(r)$ on $\mathcal{A}(D^-(0, R))$ by $|f|(r) = \sup_{n\geq 0} |a_n| r^n$ for every function $f(x) = \sum_{n\geq 0} a_n x^n$ of $\mathcal{A}(D^-(0, R))$. This norm is extended to $\mathcal{M}(D^-(0, R))$ as follows: if $f \in \mathcal{M}(D^-(0, R))$ is given by $f = \frac{g}{h}$, with $g,\ h \in \mathcal{A}(D^-(0, R))$, we write

$$|f|(r) = \left|\frac{g}{h}\right|(r) = \frac{|g|(r)}{|h|(r)}.$$

Finally, for every $f \in \mathcal{M}(D^-(0, R)) \setminus \{0\}$ and every $\alpha \in D^-(0, R)$, we denote by $\omega_\alpha(f)$ the integer i_α of $\mathbb{Z}$ such that $f(x) = \sum_{i\geq i_\alpha} a_i (x-\alpha)^i$ and $a_{i_\alpha} \neq 0$.

The following proposition is well known (see [11] for instance).

Proposition 1.1. *Let $R > 0$ and let $f \in \mathcal{M}(D^-(0, R))$ be such that 0 is neither a zero nor a pole of f. Then, for every $r \in\ ]0, R[$, we have*

$$\log |f|(r) = \log |f(0)| + \sum_{|\alpha|\leq r} \omega_\alpha(f) \log \frac{r}{|\alpha|}.$$

Let $f \in \mathcal{M}(D^-(0, R))$ be such that 0 is neither a zero nor a pole of f. For every $r \in\ ]0, R[$, we denote by $Z(r, f)$ the counting function of zeros of f in the disk $D(0, r)$, counting multiplicity, i.e., we set $Z(r, f) = \sum_{\substack{\omega_\alpha(f)>0 \\ |\alpha|\leq r}} \omega_\alpha(f) \log \frac{r}{|\alpha|}$. In the same way, we set $N(r, f) = Z(r, \frac{1}{f})$ to denote the counting function of poles of f in $D(0, r)$, counting multiplicity. Using the notation $\log^+(x) = \max(0, \log x)$, (where $x > 0$ and log is the real logarithm function), we put for $r \in\ ]0, R[$: $m(r, f) = \log^+ |f|(r)$. We finally set:

$$T(r, f) = N(r, f) + m(r, f).$$

The function $r \to T(r, f)$ is called the *Nevanlinna function* or *characteristic function* of f.