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Probabilistic properties of the relative tensor degree of finite groups

Peyman Niroomand[a], Francesco G. Russo[b,*]

[a] School of Mathematics and Computer Science, Damghan University, P.O. Box 36715–364, Damghan, Iran
[b] Department of Mathematics and Applied Mathematics, University of Cape Town, Private Bag X1, Rondebosch 7701, Cape Town, South Africa

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Abstract

Denoting by $H \otimes K$ the nonabelian tensor product of two subgroups H and K of a finite group G, we investigate the relative tensor degree $d^{\otimes}(H, K) = \frac{|\{(h,k)\in H\times K \mid h\otimes k=1\}|}{|H|\,|K|}$ of H and K. The case $H = K = G$ has been studied recently. Here we deal with arbitrary subgroups H and K, showing analogies and differences between $d^{\otimes}(H, K)$ and the relative commutativity degree $d(H, K) = \frac{|\{(h,k)\in H\times K \mid [h,k]=1\}|}{|H|\,|K|}$, which is a generalization of the probability of commuting elements, introduced by Erdős.

Keywords: Relative tensor degree; Commutativity degree; Exterior degree

1. Brown's terminology for nonabelian tensor products

We will consider finite groups only. Brown and others studied the nonabelian tensor products of groups in two classical works [2,3] almost thirty years ago. There has been a wide production in algebra and topology after these fundamental papers, because they have shown interesting relations between various areas of pure mathematics. In the context of the nonabelian tensor

* Corresponding author.
E-mail addresses: p_niroomand@yahoo.com (P. Niroomand), francescog.russo@yahoo.com (F.G. Russo).

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products, the well known notion of *Schur multiplier of a group* may be generalized to that of *Schur multiplier of a triple of groups*.

Following [4, Section 6], a *triple* (G, H, K) is a group G with two normal subgroups H and K and the Schur multiplier of (G, H, K) is an abelian group denoted by $M(G, H, K)$ and defined in terms of the *mapping cone* $B(G, H, K)$ of the *canonical cofibration* $B(G, K) \to B(G/K, HK/H)$. These notions involve some homological algebra and are defined via exact sequences in [4, (22) and (23), p. 368]. We refer in fact to [4] for feedback on mapping cone, canonical cofibration and Schur multiplier of a triple of a group.

From [2,3], a group G acts by conjugation on its normal subgroups H and K via the rule ${}^g x = gxg^{-1}$, for g in G and x in H (or K), and the *nonabelian tensor product* $H \otimes K$ is defined as the group generated by the symbols $h \otimes k$, subject to the relations:

$$h_1h_2 \otimes k_1 = ({}^{h_1}h_2 \otimes {}^{h_1}k_1)\,(h_1 \otimes k_1) \quad \text{and} \quad k_1k_2 \otimes h_1 = (k_1 \otimes h_1)\,({}^{k_1}h_1 \otimes {}^{k_1}k_2),$$

where $h_1, h_2 \in H$ and $k_1, k_2 \in K$. Adding the relation $a \otimes a = 1$ for all $a \in H \cap K$, we have the *nonabelian exterior product* $H \wedge K$ of H and K. This can be seen equivalently by introducing the *diagonal subgroup*

$$\nabla(H \cap K) = \langle a \otimes a \mid a \in H \cap K \rangle$$

and noting that

$$H \otimes K/\nabla(H \cap K) = H \wedge K.$$

In particular, we denote $\nabla(H \cap K)$ by $\nabla(G)$, when $H = K = G$. On another hand, the map

$$\kappa' : h \wedge k \in H \wedge K \mapsto \kappa'(h \wedge k) = [h, k] = hkh^{-1}k^{-1} \in [H, K]$$

is an epimorphism of groups such that

$$\ker \kappa' \simeq M(G, H, K),$$

whenever $G = HK$ and this is a very useful way to look at $M(G, H, K)$ (see [4, Theorem 6.1]). Even the map

$$\kappa : h \otimes k \in H \otimes K \mapsto \kappa(h \otimes k) = [h, k] \in [H, K]$$

is an epimorphism of groups such that $\ker \kappa$ is an abelian group, denoted by $J(G, H, K)$. If $G = HK$ (with H and K normal in G), then the following diagram is commutative

$$\begin{array}{ccccccccc} 1 & \longrightarrow & J(G,H,K) & \longrightarrow & H \otimes K & \xrightarrow{\kappa} & [H,K] & \longrightarrow & 1 \\ & & \pi\downarrow & & \varepsilon\downarrow & & \| & & \\ 1 & \longrightarrow & M(G,H,K) & \longrightarrow & H \wedge K & \xrightarrow{\kappa'} & [H,K] & \longrightarrow & 1 \end{array} \qquad (*)$$

with central extensions as rows and natural epimorphisms

$$\pi : h \otimes k \in J(G, H, K) \mapsto h \wedge k \in M(G, H, K),$$
$$\varepsilon : h \otimes k \in H \otimes K \mapsto h \wedge k \in H \wedge K$$

as columns. Of course, if $G = H = K$, we have that $M(G) = H_2(G, \mathbb{Z})$ is exactly the *Schur multiplier* of G (i.e.: the second group of homology with integral coefficients on G), $G \otimes G$ is the *nonabelian tensor square* of G and $G \wedge G$ is the *nonabelian exterior square* of G. Some