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Indagationes Mathematicae 27 (2016) 160–172

indagationes mathematicae

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Algebraic independence of subseries over arithmetical progressions of certain exponential power series

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Received 13 February 2015; received in revised form 5 September 2015; accepted 15 September 2015

Communicated by R. Tijdeman

Abstract

Let $q \geq 2$ be an integer. We separate a given power series into q subseries according to the residue classes mod q of their powers. We study algebraic independence for values at an algebraic point of the q subseries of certain exponential power series including the exponential generating function of Fibonacci numbers. Our proofs rely on Lindemann–Weierstrass theorem.

Keywords: Algebraic independence; Subseries mod q; Exponential generating functions; Fibonacci numbers; Lindemann–Weierstrass theorem

1. Introduction

Let q be a positive integer. For a given series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{C}[[z]],$$

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http://dx.doi.org/10.1016/j.indag.2015.09.003
0019-3577/

we consider the subseries over arithmetical progressions

$$f_{q,r}(z) = \sum_{\substack{n=0 \\ n\equiv r \pmod{q}}}^{\infty} a_n z^n = \sum_{k=0}^{\infty} a_{qk+r} z^{qk+r} \quad (0 \le r < q), \tag{1}$$

which are called for short *subseries mod q* (see Section 2). The subseries mod q of the exponential function e^z are

$$e_{q,r}(z) := \sum_{\substack{n=0 \\ n\equiv r \pmod{q}}}^{\infty} \frac{z^n}{n!} = \frac{1}{q} \sum_{k=0}^{q-1} \xi^{-kr} e^{\xi^k z} \quad (0 \le r < q),$$

where $\xi = e^{2\pi i/q}$ (see (13)). If α is a nonzero algebraic number, then each of the numbers $e_{q,r}(\alpha) \quad (0 \le r < q)$ is transcendental by Lindemann–Weierstrass theorem (cf. [8]). Recently C. Elsner, Yu. V. Nesterenko, and the second named author proved the following:

Theorem 1.1 ([1, Theorem 1]). *Let $q \ge 3$ be an integer. If α is a nonzero algebraic number, then among q numbers*

$$e_{q,0}(\alpha),\ e_{q,1}(\alpha), \dots,\ e_{q,q-1}(\alpha)$$

any $\varphi(q)$ are algebraically independent over $\mathbb{Q}$, where φ is the Euler totient. Moreover, any $\varphi(q)+1$ of q functions $e_{q,0}(z),\ e_{q,1}(z), \dots,\ e_{q,q-1}(z)$ are algebraically dependent over $\mathbb{Q}$.

In this paper, we investigate algebraic independence over $\mathbb{Q}$ of the values at an algebraic point of subseries mod q of the series

$$\sum_{n=0}^{\infty} \frac{a_1\rho_1^n + a_2\rho_2^n}{n!} z^n = a_1 e^{\rho_1 z} + a_2 e^{\rho_2 z}, \tag{2}$$

where a_1, a_2, ρ_1, ρ_2 are algebraic numbers different from zero. We obtain the following results.

Theorem 1.2. *Let $q \ge 2$ be an integer and a_1, a_2, ρ_1, ρ_2 be nonzero algebraic numbers. Assume that $\rho_1/\rho_2 \notin \mathbb{Q}(\xi)$, where $\xi = e^{2\pi i/q}$. If α is a nonzero algebraic number, then among q numbers*

$$\sum_{\substack{n=0 \\ n\equiv r \pmod{q}}} \frac{a_1\rho_1^n + a_2\rho_2^n}{n!} \alpha^n \quad (0 \le r < q) \tag{3}$$

any $\min\{2\varphi(q), q\}$ *are algebraically independent over $\mathbb{Q}$ and not are any* $\min\{2\varphi(q), q\} + 1$.

If $\rho_1/\rho_2 \in \mathbb{Q}(\xi)$, then some of the subseries mod q given in (3) may be zero. For example, let $a_1 = a_2 = 1$ and $\rho_2 = -\rho_1$. Then the subseries with $q = 2, r = 1$ and also those with $q = 4$, $\rho_1 = i, r = 1$ or 3 vanish.

Let $\{F_n\}_{n\ge0}$ and $\{L_n\}_{n\ge0}$ be Fibonacci and Lucas numbers defined by $F_0 = 0, F_1 = 1, F_{n+1} = F_n + F_{n-1}, L_0 = 2, L_1 = 1, L_{n+1} = L_n + L_{n-1}$ $(n \ge 1)$. Specializing the function (2) to the exponential generating function for Fibonacci numbers (cf. e.g., [4])

$$\sum_{n=0}^{\infty} \frac{F_n}{n!} z^n = \frac{2}{\sqrt{5}} e^{z/2} \sinh(\sqrt{5}z/2), \tag{4}$$

we deduce