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The exponential law for spaces of test functions and diffeomorphism groups

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Abstract

We prove the exponential law $\mathcal{A}(E \times F, G) \cong \mathcal{A}(E, \mathcal{A}(F, G))$ (bornological isomorphism) for the following classes $\mathcal{A}$ of test functions: $\mathcal{B}$ (globally bounded derivatives), $W^{\infty,p}$ (globally p-integrable derivatives), $\mathcal{S}$ (Schwartz space), $\mathcal{D}$ (compact support), $\mathcal{B}^{[M]}$ (globally Denjoy–Carleman), $W^{[M],p}$ (Sobolev–Denjoy–Carleman), $\mathcal{S}_{[L]}^{[M]}$ (Gelfand–Shilov), and $\mathcal{D}^{[M]}$ (Denjoy–Carleman with compact support). Here E, F, G are convenient vector spaces which are finite dimensional in the cases of $\mathcal{D}$, $W^{\infty,p}$, $\mathcal{D}^{[M]}$, and $W^{[M],p}$. Moreover, $M = (M_k)$ is a weakly log-convex weight sequence of moderate growth. As application we give a new simple proof of the fact that the groups of diffeomorphisms $\operatorname{Diff}\mathcal{B}$, $\operatorname{Diff} W^{\infty,p}$, $\operatorname{Diff}\mathcal{S}$, and $\operatorname{Diff}\mathcal{D}$ are C^∞ Lie groups, and that $\operatorname{Diff}\mathcal{B}^{\{M\}}$, $\operatorname{Diff} W^{\{M\},p}$, $\operatorname{Diff}\mathcal{S}_{\{L\}}^{\{M\}}$, and $\operatorname{Diff}\mathcal{D}^{\{M\}}$, for non-quasianalytic M, are $C^{\{M\}}$ Lie groups, where $\operatorname{Diff}\mathcal{A} = \{\mathrm{Id} + f : f \in \mathcal{A}(\mathbb{R}^n, \mathbb{R}^n), \inf_{x\in\mathbb{R}^n} \det(\mathbb{I}_n + df(x)) > 0\}$. We also discuss stability under composition.

Keywords: Convenient setting; Exponential law; Test functions; Sobolev functions; Denjoy–Carleman classes; Gelfand–Shilov classes

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1. Introduction

In this paper we prove the bornological isomorphism

$$\mathcal{A}(E \times F, G) \cong \mathcal{A}(E, \mathcal{A}(F, G)) \tag{1}$$

for several classes $\mathcal{A}$ of test functions. It is called *exponential law*, since it takes the form $G^{E\times F} = (G^F)^E$ if one writes $\mathcal{A}(X, Y) = Y^X$.

The exponential law (1) is well-known in the categories of C^∞, real analytic, and holomorphic functions; see [8]. In [9–11] we established the exponential law (1) for local Denjoy–Carleman classes $C^{[M]}$, provided that $M = (M_k)$ is weakly log-convex and has moderate growth. (The notation $C^{[M]}$ stands for the classes $C^{\{M\}}$ of Roumieu type as well as for the classes $C^{(M)}$ of Beurling type, cf. Section 2.2.) In all these cases the underlying spaces E, F, G are so-called *convenient* vector spaces, i.e., locally convex spaces that are Mackey complete.

We shall prove (1) for the following classes $\mathcal{A}$ of test functions (see Sections 3 and 6 for the precise definitions):

- Smooth functions with globally bounded derivatives $\mathcal{B}$ (=$\mathcal{D}_{L^\infty}$ in [19])
- Smooth functions with p-integrable derivatives $W^{\infty,p}$ (=$\mathcal{D}_{L^p}$ in [19])
- Rapidly decreasing Schwartz functions $\mathcal{S}$
- Smooth functions with compact support $\mathcal{D}$
- Global Denjoy–Carleman classes $\mathcal{B}^{[M]}$
- Sobolev–Denjoy–Carleman classes $W^{[M],p}$
- Gelfand–Shilov classes $\mathcal{S}_{[L]}^{[M]}$
- Denjoy–Carleman functions with compact support $\mathcal{D}^{[M]}$

For the sequence $L = (L_k)$ we just assume $L_k \geq 1$ for all k.

The underlying spaces are again convenient vector spaces, except for $\mathcal{D}$, $W^{\infty,p}$, $\mathcal{D}^{[M]}$, and $W^{[M],p}$ when E, F, G are assumed to be finite dimensional. The definition of the classes $\mathcal{B}$, $\mathcal{S}$, $\mathcal{B}^{[M]}$, and $\mathcal{S}_{[L]}^{[M]}$ makes obvious sense between arbitrary Banach spaces. By definition, a C^∞-mapping $f : E \to F$ between general convenient vector spaces belongs to the class if the composite $\ell \circ f \circ i_B : E_B \to \mathbb{R}$ is in the class for each continuous linear functional $\ell : F \to \mathbb{R}$ and each closed absolutely convex bounded subset $B \subseteq E$, where $i_B : E_B \to E$ denotes the inclusions of the linear span E_B of B which equipped with the Minkowski functional is a Banach space.

For finite dimensional parameter spaces we have the following continuous inclusions, where $1 \leq p < q < \infty$; for the inclusions marked by $*$ we assume that $M = (M_k)$ is derivation closed.

$$\begin{tikzcd}
\mathcal{D} \arrow[r, tail] & \mathcal{S} \arrow[r, tail] & W^{\infty,p} \arrow[r, tail] & W^{\infty,q} \arrow[r, tail] & \mathcal{B} \arrow[r, tail] & C^\infty \\
\mathcal{D}^{\{M\}} \arrow[r, tail] \arrow[u, tail] & \mathcal{S}_{\{L\}}^{\{M\}} \arrow[r, tail] \arrow[u, tail] & W^{\{M\},p} \arrow[r, tail, "*"] \arrow[u, tail] & W^{\{M\},q} \arrow[r, tail, "*"] \arrow[u, tail] & \mathcal{B}^{\{M\}} \arrow[r, tail] \arrow[u, tail] & C^{\{M\}} \arrow[u, tail] \\
\mathcal{D}^{(M)} \arrow[r, tail] \arrow[u, tail] & \mathcal{S}_{(L)}^{(M)} \arrow[r, tail] \arrow[u, tail] & W^{(M),p} \arrow[r, tail, "*"] \arrow[u, tail] & W^{(M),q} \arrow[r, tail, "*"] \arrow[u, tail] & \mathcal{B}^{(M)} \arrow[r, tail] \arrow[u, tail] & C^{(M)} \arrow[u, tail]
\end{tikzcd}$$

We are grateful to a referee who pointed out that

$$\mathcal{D}(\mathbb{R}^\ell \times \mathbb{R}^m, \mathbb{R}^n) \cong \mathcal{D}(\mathbb{R}^\ell, \mathcal{D}(\mathbb{R}^m, \mathbb{R}^n)) \tag{2}$$