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Sharp symplectic embeddings of cylinders

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Abstract

We show that the cylinder $\mathrm{B}^2(1) \times \mathbb{R}^{2(n-1)}$ embeds symplectically into $\mathrm{B}^4(\sqrt{3}) \times \mathbb{R}^{2(n-2)}$.

Keywords: Symplectic topology; Embedding; Symplectic capacity

1. Introduction

The existence (or not) of symplectic embeddings has been a driving question in symplectic topology since Gromov's pioneering article [3], and there have been a number of important papers in the subject in the recent years, see for instance Biran [1], Hind–Kerman [5], Lalonde–Pinsonnault [6], McDuff–Polterovich [7], McDuff–Schlenk [8], and Schlenk [10]. In this paper our ambient space is $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$ on which we consider the standard symplectic form $\mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_n \wedge \mathrm{d}y_n$. A smooth embedding $F : U \to V$ between subsets of $\mathbb{R}^{2n}$ is *symplectic* if F pulls back this form to itself. Let $\mathrm{B}^{2n}(R)$ denote the

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open ball of radius R in $\mathbb{R}^{2n}$, where $R > 0$, that is, the set of points $(x_1, y_1 \ldots, x_n, y_n) \in \mathbb{R}^{2n}$ such that $\sum_{i=1}^{n}(x_i)^2 + (y_i)^2 < R^2$.

Question 1.1 (*Hind and Kerman [5, Question 2]*)**.** Let $n \geqslant 3$. *Can* $\overline{\mathrm{B}^2(1)} \times \mathrm{B}^{2(n-1)}(S)$ *be symplectically embedded into* $\overline{\mathrm{B}^4(R)} \times \mathbb{R}^{2(n-2)}$ *for arbitrarily large* $S > 0$*? If so, what is the smallest* $R > 0$ *for which this is possible?*

In [4], Guth proved that such an R exists, which was a surprise at that time. Then, the question was settled by Hind–Kerman [5] for all numbers R but one: $R = \sqrt{3}$. They proved that there are embeddings when $R > \sqrt{3}$ for all $S > 0$, but there are not such embeddings if $R < \sqrt{3}$ and S is sufficiently large. Prior to their work it was known that the Ekeland–Hofer capacity implied $R > \sqrt{2}$, if such embeddings did exist (see [2]).

Let $\mathrm{Z}^{2n}(r)$ denote the cylinder of radius r in $\mathbb{R}^{2n}$, where $r > 0$, that is, the set of points $(x_1, y_1 \ldots, x_n, y_n) \in \mathbb{R}^{2n}$ such that $(x_1)^2 + (y_1)^2 < r^2$. The goal of this paper is to show the following theorem about symplectic embeddings of cylinders, which in particular completes the answer to Question 1.1 in the *case of open balls* by answering the end-point case.

Theorem 1.2. *The cylinder* $\mathrm{Z}^{2n}(1)$ *embeds symplectically into the product* $\mathrm{B}^4(\sqrt{3}) \times \mathbb{R}^{2(n-2)}$.

It follows from Hind–Kerman [5] and Theorem 1.2 that the cylinder $\mathrm{Z}^{2n}(1)$ embeds symplectically into the product $\mathrm{B}^4(R) \times \mathbb{R}^{2(n-2)}$ if and only if $R \geqslant \sqrt{3}$. The proof of Theorem 1.2 relies on [4,9] and follows closely essential ideas of [5].

Remark 1.3. Theorem 1.2 can be used to derive an alternative proof of the inexistence of symplectic d-capacities ($1 < d < n$) proven in [9]. ⊘

2. Smoothness of families and Guth's Lemma

Following [9, Section 3], we have the following natural notions.

Definition 2.1. Let P, N be smooth manifolds. Let $(B_p)_{p \in P}$ be a family of submanifolds of N. We say that $(B_p)_{p \in P}$ is a *smooth family* if there is a smooth manifold B and a smooth map $g : P \times B \to N$ such that $g_p : b \mapsto g(p, b)$ is an embedding and $B_p = g(p, B)$, for every $p \in P$. The family is a *smooth family of symplectic manifolds* if each B_p is a symplectic submanifold with respect to the symplectic structure inherited by B_p from N.

Definition 2.2. Let P, M, N be smooth manifolds. Let $(B_p)_{p \in P}$ be a family of submanifolds of N. For each $p \in P$, let $\phi_p : B_p \hookrightarrow M$ be an embedding. We say that $(\phi_p)_{p \in P}$ is a *smooth family of embeddings* if the following properties hold:

(1) the family $(B_p)_{p \in P}$ is a smooth family of submanifolds in the sense of Definition 2.1;
(2) the map $\Phi : P \times B \to M$ defined by $\Phi(p, b) := \phi_p \circ g(p, b)$ is smooth.

In this case we also say that $(\phi_p \colon B_p \hookrightarrow M_p)_{p \in P}$ is a *smooth family of embeddings* when M_p is a submanifold of M containing $\phi_p(B_p)$. If M and N are symplectic, then a *smooth family of symplectic embeddings* is a smooth family of embeddings $(\phi_p)_{p \in P}$ such that each B_p is a symplectic submanifold and $\phi_p : B_p \hookrightarrow M$ is symplectic.

Definition 2.3. If in Definition 2.1 or in Definition 2.2 P is a subset of a smooth manifold $\tilde{P}$, then we say that the families $(B_p)_{p \in P}$ or $(\phi_p)_{p \in P}$ are *smooth* if there is an open neighborhood U of P such that $g : P \times B \to N$ or $\Phi : P \times B \to M$ may be smoothly extended to $U \times B$.