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Diophantine equations with truncated binomial polynomials

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Abstract

For positive integers $k \leq n$ let $P_{n,k}(x) := \sum_{j=0}^{k} \binom{n}{j} x^j$ be the binomial expansion of $(1+x)^n$ truncated at the kth stage. In this paper we show the finiteness of solutions of Diophantine equations of type $P_{n,k}(x) = P_{m,l}(y)$ in $x, y \in \mathbb{Z}$ under assumption of irreducibility of truncated binomial polynomials $P_{n-1,k-1}(x)$ and $P_{m-1,l-1}(x)$. Although the irreducibility of $P_{n,k}(x)$ has been studied by several authors, in general, this problem is still open. In addition, we give some results about the possible ways to write $P_{n,k}(x)$ as a functional composition of two lower degree polynomials.

Keywords: Truncated binomial expansion; Dickson polynomial; Diophantine equations

1. Introduction and main results

For positive integers $k \leq n$ put

$$P_{n,k}(x) := \sum_{j=0}^{k} \binom{n}{j} x^j = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{k} x^k.$$

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The polynomial $P_{n,k}(x)$ is said to be a *truncated binomial expansion* (*polynomial*) at the kth stage. We study Diophantine equations of type

$$P_{n,k}(x) = P_{m,l}(y) \quad \text{with } n, k, m, l \in \mathbb{N},\ k \le n-1,\ l \le m-1. \tag{1.1}$$

We prove that Eq. (1.1) has only finitely many integer solutions under certain reasonable assumptions. The main results are deduced from a general finiteness criterion for the Diophantine equation $f(x) = g(y)$ established by Bilu and Tichy in [3]. The proof requires several auxiliary results about the possible ways to write a truncated binomial expansion as a functional composition of two lower degree polynomials, as the above mentioned theorem of Bilu and Tichy essentially says that the equation of type $f(x) = g(y)$ has only finitely many solutions in integers x, y, unless the polynomials $f(x)$ and $g(x)$ can be written as a functional composition of some lower degree polynomials in a prescribed way. Factorization of polynomials under the operation of functional composition, i.e. "polynomial decomposition" was first studied by Ritt [13], and subsequently investigated and applied by many other authors; see, for instance, [3,6,9,11,12,15,18,20].

Our interest in Eq. (1.1) has arisen from our considerations of decomposition properties of truncated binomial expansions. Note that $P'_{n,k}(x) = nP_{n-1,k-1}(x)$, so if $P_{n,k}(x) = g(h(x))$, where $g(x), h(x) \in \mathbb{Q}[x]$ satisfy $\deg g > 1$ and $\deg h > 1$, then

$$nP_{n-1,k-1}(x) = P'_{n,k}(x) = g'(h(x))h'(x),$$

and, consequently, the polynomial $P_{n-1,k-1}(x)$ is reducible over $\mathbb{Q}$. The question of irreducibility of truncated binomial expansions first appeared in [14]. It was studied by Filaseta, Kumchev and Pasechnik [8], and subsequently by Khanduja, Khassa and Laishram [10]. There are indications that the polynomials $P_{n,k}(x)$ are irreducible for all pairs $k, n \in \mathbb{N}$ satisfying $k \le n-2$, although this problem is still far from being solved. It is known that $P_{n,k}(x)$ are irreducible for $n \le 100$ and $k \le n-2$, see [8]. It is easy to see that $P_{n,k}(x)$ is irreducible for $k = 2$, since in this case the discriminant of the polynomial is negative, so that it has two complex roots. It is also known that $P_{n,k}(x)$ are irreducible for all $k, n \in \mathbb{N}$ satisfying $2k \le n < (k+1)^3$, see [10]. Furthermore, as it was shown in [8] for any fixed integer $k \ge 3$ there exists an integer $n_0(k)$ such that $P_{n,k}(x)$ is irreducible for every $n \ge n_0$. Finally, in the same paper it was proved that if n is prime, then $P_{n,k}(x)$ is irreducible for each k in the range $1 \le k \le n-1$.

In this paper we prove the following.

Theorem 1.1. *Let $n, k, m, l \in \mathbb{N}$ be such that $2 \le k \le n-1, 2 \le l \le m-1$ and $k \ne l$. If $P_{n-1,k-1}(x)$ and $P_{m-1,l-1}(x)$ are irreducible, then the equation $P_{n,k}(x) = P_{m,l}(y)$ has at most finitely many integer solutions (x, y).*

Note that the truncated binomial expansion at the last stage, i.e. when $k = n-1$, takes the form $P_{n,n-1}(x) = (x+1)^n - x^n$. If n is a composite integer, then $P_{n,n-1}(x)$ is clearly reducible. If $n = p$ is a prime, then the polynomial $P_{n,n-1}(x) = P_{p,p-1}(x)$ is irreducible, by the Eisenstein criterion applied to the reciprocal polynomial $x^{p-1}P_{p,p-1}(1/x)$. As an auxiliary result we show that if n is even, then $P_{n,n-1}(x)$ cannot be written in the form

$$P_{n,n-1}(x) = g(x) \circ h(x) = g(h(x))$$

with $g(x), h(x) \in \mathbb{C}[x]$ and $\deg g > 1$, $\deg h > 1$. We further show that if n is odd, then essentially the only way to write $P_{n,n-1}(x)$ as a functional composition of polynomials of lower degree is the following: write $n = 2n' + 1$ and $\omega_j = \exp(2\pi ij/n)$, $j = 1, 2, \ldots, n$, so that