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Asymptotic behavior and stability of second order neutral delay differential equations

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Abstract

We study the asymptotic behavior of a class of second order neutral delay differential equations by both a spectral projection method and an ordinary differential equation method approach. We discuss the relation of these two methods and illustrate some features using examples. Furthermore, a fixed point method is introduced as a third approach to study the asymptotic behavior. We conclude the paper with an application to a mechanical model of turning processes.

Keywords: Asymptotic behavior; Stability; Asymptotic stability; Neutral delay differential equation; Retarded delay differential equation; Characteristic equation; Fixed point theory; Spectral theory

1. Introduction

Neutral delay differential equations arise from a variety of applications including control systems, electrodynamics, mixing liquids, neutron transportation and population models. In the qualitative analysis of such systems, the stability and asymptotic behavior of solutions play an important role. In 1973, Driver, Sasser and Slater [4] studied asymptotic behavior, oscillation and stability of first order delay differential equations with small delay using an approach based on an

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ordinary differential equation (ODE) method. The key idea of the ODE approach is to transform the differential equation into a lower order equation by using a real root of the corresponding characteristic equation. Following this approach as presented in [4], a number of papers appeared in which the asymptotic behavior, oscillation and stability for first (or second or higher) order (neutral) delay differential equations, and integro-differential equations with unbounded delay as well as for delay difference equations were studied, see [9,15,14,13,12]. A disadvantage of this ODE approach is that it does not lead to explicit formulas for the reduced lower order equations.

In 2003, by using spectral theory, Frasson and Verduyn Lunel [7] presented a new approach to study the asymptotic behavior of neutral delay differential equations, the so-called spectral projection method.

In this paper it is our intention to compare the two approaches. We discuss their relations by studying asymptotic behavior of a class of second order neutral delay differential equations. We obtain that under the same assumptions, the ODE approach is equivalent to the spectral approach (see Section 4). However, the spectral approach has some advantages, since the conditions for the spectral method are weaker than those needed for the ODE method, as is illustrated by Example 4.6, and the asymptotic behavior of neutral delay differential equations can be presented by a general formula (see Theorem 2.5). Furthermore, by using the spectral approach, we can also study the asymptotic behavior of neutral delay differential equations with matrix coefficients.

We consider a specific class of second order neutral delay differential equations of the following form

$$\begin{cases} x''(t) + cx''(t-\tau) = p_1 x'(t) + p_2 x'(t-\tau) + q_1 x(t) + q_2 x(t-\tau), \\ x(t) = \phi(t), \quad -\tau \le t \le 0, \end{cases} \tag{1}$$

where $c, p_1, p_2, q_1, q_2 \in \mathbb{R}$, $\tau > 0$, the initial function ϕ is a given continuously differentiable real-valued function on the initial interval $[-\tau, 0]$.

Such neutral delay differential equations arise, for example, in the study of interconnected oscillatory systems described by a system of linear hyperbolic partial differential equations. Here the motion of each of the individual oscillatory systems is described by a boundary condition and the interconnection is given by a traveling wave. Generally, such systems can be transformed into delay equations involving delays in the highest derivative, see [9]. Furthermore, neutral delay differential equations serve as population models in the sense that they can be interpreted as special cases of the standard Gurtin–MacCamy model for a population structured by age with birth and death rates depending on the total adult population, see [8].

Eq. (1) is a general second order neutral differential equation with constant coefficients, where all delays in the delayed terms are assumed to be equal and the coefficient in front of $x''(t)$ is assumed to be nonzero and scaled to 1. The second order equation is rich enough to illustrate the analysis of equations of order higher than one and simple enough to allow for a clear comparison of different approaches.

A special case of system (1) is a retarded delay equation, i.e.,

$$x''(t) + ax'(t) + bx(t-r) + cx(t) = 0, \quad a, b, c \in \mathbb{R}, \ r > 0, \tag{2}$$

which is often called a delayed oscillator, is well-studied in applications. It appears, for example, as the basic governing equation of the regenerative model of machine tool chatter. We illustrate a third approach, based on a fixed point method, to study the asymptotic behavior of such equations.