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A metric discrepancy result for the sequence of powers of minus two

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Abstract

The law of the iterated logarithm for discrepancies of $\{(-2)^k t\}_k$ is proved. This result completes the concrete determination of the law of the iterated logarithm for discrepancies of the geometric progression with integer ratio, and reveals the fact that 2 is the only positive integer $\theta > 1$ such that fractional parts of $\{(-\theta)^k t\}_k$ converge to uniform distribution faster than those of $\{\theta^k t\}_k$ a.e. t.

Keywords: Discrepancy; Lacunary sequence; Law of the iterated logarithm

1. Introduction

Kronecker [22] proved that the sequence of the fractional part of kt ($k = 1, 2, \dots$) is dense in the unit interval if and only if t is irrational, and it was more than twenty years later that Bohl [7], Sierpiński [27] and Weyl [30] proved independently that the sequence is uniformly distributed modulo one in the following sense: a sequence $\{x_k\}$ of real numbers is said to be uniformly distributed modulo one if ${}^{\#}\{k \le N \mid \langle x_k \rangle \in [a, b)\}/N \to b - a$ for all $[a, b) \subset [0, 1)$, where $\langle x \rangle$ denotes the fractional part $x - [x]$ of real number x. These results initiated the theory of uniform distribution.

We use the following discrepancies $D_N\{x_k\}$ and $D_N^*\{x_k\}$ to measure the speed of convergence (see [10]):

$$D_N\{x_k\} = \sup_{0 \le a < b < 1} \left| \frac{1}{N} {}^{\#}\{k \le N \mid \langle x_k \rangle \in [a, b)\} - (b - a) \right|,$$

$$D_N^*\{x_k\} = \sup_{0 \le a < 1} \left| \frac{1}{N} {}^{\#}\{k \le N \mid \langle x_k \rangle \in [0, a)\} - a \right|.$$

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Weyl proved $D_N^*\{n_k t\} \to 0$ a.e. t under very mild condition $n_{k+1} - n_k > C > 0$ for all large k, and showed that the method of measure theory is effective in the research of the uniform distribution theory.

Various studies were done in this direction. For arithmetic progressions $\{kt\}$ and increasing functions g, Khintchine [21] proved that

$$ND_N^*\{kt\} = O\left((\log N)g(\log\log N)\right) \quad \text{a.e. } t$$

holds if and only if the function g satisfies $\sum 1/g(n) < \infty$. When $\sum 1/g(n) < \infty$ is satisfied, we can easily derive a stronger result

$$ND_N^*\{kt\} = o\left((\log N)g(\log\log N)\right) \quad \text{a.e. } t,$$

and see that critical speed cannot be determined in almost everywhere sense. The critical speed was determined by Kesten [20] in the sense of convergence in measure:

$$\lim_{N\to\infty} \text{Leb}\left\{t \in [0,1) \,\middle|\, \left|\frac{ND_N^*\{kt\}}{\log N \log\log N} - \frac{2}{\pi^2}\right| > \varepsilon\right\} = 0, \quad (\varepsilon > 0).$$

In probability theory, the following beautiful result was proved by Chung [8] and Smirnov [28] independently, viz. the law of the iterated logarithm

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{U_k\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{U_k\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.s.}$$

where $\{U_k\}$ is the sequence of independent and uniformly distributed random variables.

After a number of studies on the behavior of $D_N\{n_k t\}$ for increasing $\{n_k\}$, Erdős [11] conjectured $ND_N\{n_k t\} = O((N\log\log N)^{1/2})$ a.e. assuming the Hadamard gap condition $n_{k+1}/n_k \geq q > 1$. Since the law of the iterated logarithm

$$\varlimsup_{N\to\infty} \frac{1}{\sqrt{2N\log\log N}} \sum_{k=1}^{N} \cos 2\pi n_k t = \frac{1}{\sqrt{2}} \quad \text{a.e. } t$$

was proved under the Hadamard gap condition by Erdős–Gál [12], it was natural to expect the analogue of the Chung–Smirnov result above.

By using Takahashi's method [29], Philipp [24] solved the conjecture by showing the bounded law of the iterated logarithm

$$\frac{1}{4\sqrt{2}} \leq \varlimsup_{N\to\infty} \frac{ND_N^*\{n_k t\}}{\sqrt{2N\log\log N}} \leq \varlimsup_{N\to\infty} \frac{ND_N\{n_k t\}}{\sqrt{2N\log\log N}} \leq \frac{1}{\sqrt{2}}\left(166 + \frac{664}{q^{1/2}-1}\right) \quad \text{a.e. } t.$$

For a proof using martingales and another approach, see Philipp [23,25]. Dhompongsa [9] assumed the very strong gap condition

$$\log(n_{k+1}/n_k)/\log\log k \to \infty \quad (k \to \infty)$$

and derived the Chung–Smirnov type result

$$\varlimsup_{N\to\infty} \frac{ND_N^*\{n_k t\}}{\sqrt{2N\log\log N}} = \varlimsup_{N\to\infty} \frac{ND_N\{n_k t\}}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.e. } t.$$

The condition was relaxed later [1,14] to $n_{k+1}/n_k \to \infty$.