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Analytic and algebraic properties of Riccati equations: A survey

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Abstract

This is a survey of recent results on the classical problems of the *analytic properties of Riccati equations* and *algebraic properties of Riccati equations* and applications to spatially distributed systems.

Keywords: Linear systems; Infinite-dimensional systems; Spatially invariant systems; Platoons; Spatially distributed systems

1. Motivation: spatially distributed systems

In this article we consider two classical problems for solutions to Riccati equations:

• *Analytic property:* Suppose that $A_1(z), A_2(z), D(z), M(z)$ are analytic $n \times n$ matrix functions of the complex variable z on a connected open subset Ω of the complex plane $\mathbb{C}$. Under what conditions will the Riccati equation

$$A_1(z)Q(z) + Q(z)A_2(z) - Q(z)D(z)Q(z) + M(z) = 0,$$

have a solution $Q(z)$ that is analytic in a subset of Ω?

• *Algebraic property:* Let R be a Banach algebra with the involution $\cdot^\star$. Given $A \in R^{n\times n}$, $B \in R^{m\times n}$, $A \in R^{n\times p}$, under what conditions will the algebraic Riccati equation

$$A^\star Q + QA - QBB^\star Q + C^\star C = 0 \tag{1}$$

have a solution $Q = Q^\star$ in R?

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These problems have been considered earlier from a mathematical perspective. However, as has often been the case in systems theory, new engineering applications give rise to new mathematical formulations and new insights into the theory. We illustrate this phenomenon with the following class of systems:

$$\dot{z}_r(t) = \sum_{l\in\mathbb{Z}} A_{rl} z_l(t) + \sum_{l\in\mathbb{Z}} B_{rl} u_l(t), \tag{2}$$

$$y_r(t) = \sum_{l\in\mathbb{Z}} C_{rl} z_l(t) + \sum_{l\in\mathbb{Z}} D_{rl} u_l(t), \tag{3}$$

where $r, l \in \mathbb{Z}$, the set of integers, $A_{rl} \in \mathbb{C}^{n\times n}, B_{rl} \in \mathbb{C}^{n\times m}, C_{rl} \in \mathbb{C}^{p\times n}, D_{rl} \in \mathbb{C}^{p\times m}$ and $z_r(t) \in \mathbb{C}^n, u_r(t) \in \mathbb{C}^m$ and $y_r(t) \in \mathbb{C}^p$ are the state, the input and the output vectors, respectively, at time $t \geq 0$ and spatial point $r \in \mathbb{Z}$. This class belongs to the class of *spatially distributed systems* introduced in Bamieh et al. [1]. In fact, they form a special class of infinite-dimensional systems.

Using the terminology and formalism of Curtain and Zwart [12], Eqs. (2) and (3) can be formulated as an infinite-dimensional linear system $\Sigma(A, B, C, D)$

$$\begin{aligned} \dot{z}(t) &= (Az)(t) + (Bu)(t), \\ y(t) &= (Cz)(t) + (Du)(t), \quad t \geq 0, \end{aligned} \tag{4}$$

with the state space $Z = \ell_2(\mathbb{C}^n)$, the input space $U = \ell_2(\mathbb{C}^m)$ and the output space $Y = \ell_2(\mathbb{C}^p)$.

In the special case where $A_{rl} = A_{r-l}$ the system (2), (3) reduces to

$$\dot{z}_r(t) = \sum_{l\in\mathbb{Z}} A_l z_{r-l}(t) + \sum_{l\in\mathbb{Z}} B_l u_{r-l}(t), \tag{5}$$

$$y_r(t) = \sum_{l\in\mathbb{Z}} C_l z_{r-l}(t) + \sum_{l\in\mathbb{Z}} D_l u_{r-l}(t). \tag{6}$$

The corresponding operators A, B, C, D in the formulation (4) are convolution operators. Denoting the signals and the convolution operators generically by $x(t)$ and T, respectively, we have

$$((Tx)(t))_r = \sum_{l\in\mathbb{Z}} T_l x_{r-l}(t) = \sum_{l\in\mathbb{Z}} T_{r-l} x_l(t).$$

The T are shift-invariant operators in $\ell_2(\mathbb{C}^n)$ which in the engineering literature are called the *spatially invariant* operators. One of the motivations for studying these spatially invariant systems stems from the interest shown in the engineering literature in controlling infinite platoons of vehicles over the years [17–19,4,16]. We shall call these systems *platoon-type systems*.

Taking (formally) discrete Fourier transforms $\mathfrak{F} : \ell_2(\mathbb{C}^n) \to \mathbf{L}_2(\mathbb{T}; \mathbb{C}^n)$ of the system equations (4), where $\mathbb{T}$ denotes the unit circle, we obtain

$$\begin{aligned} \dot{\check{z}}(t) &= \mathfrak{F}\dot{z}(t) = \check{A}\check{z}(t) + \check{B}\check{u}(t), \\ \check{y}(t) &= \mathfrak{F}y(t) = \check{C}\check{z}(t) + \check{D}\check{u}(t). \end{aligned} \tag{7}$$

Note that $\check{z}$ denotes the Fourier transform $\mathfrak{F}z$ of z and $\check{A} = \mathfrak{F}A\mathfrak{F}^{-1}$, $\check{B} = \mathfrak{F}B\mathfrak{F}^{-1}$, $\check{C} = \mathfrak{F}C\mathfrak{F}^{-1}$ and $\check{D} = \mathfrak{F}D\mathfrak{F}^{-1}$ are shift-invariant operators. If $\check{A}, \check{B}, \check{C}, \check{D} \in \mathbf{L}_\infty(\mathbb{T}; \mathbb{C}^{\bullet\times\bullet})$, then A, B, C, D are all bounded operators ("$\bullet$" denotes the appropriate dimension). We shall