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The finite section method for infinite Vandermonde matrices

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Dedicated to the memory of Israel Gohberg

Abstract

The finite section method for infinite Vandermonde matrices is the focus of this paper. In particular, it is shown that for a large class of infinite Vandermonde matrices the finite section method converges in l_1 sense if the right hand side of the equation is in a suitably weighted $l_1(\alpha)$ space. Some explicit results are obtained for a wide class of examples.

Keywords: Finite section method; Infinite Vandermonde matrix; Infinite systems of equations

1. Introduction

Already in the nineteenth century are there cases in the mathematical literature where an infinite system of linear equations in an infinite number of unknowns needs to be solved. The situation gave rise to the 1913 book of Riesz [5] and in later years greatly influenced the development of functional analysis and operator theory.

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A natural approach to finding a solution of a system containing countably many equations and unknowns is the following. Take the first n equations and n unknowns, neglect the rest; then we have a finite system, which we solve. As n grows larger, we expect the solutions of the finite systems to approximate a solution of the infinite system.

This method, which is called the *finite section method*, appears already in the work of Fourier (cited in [5]). Fourier looks for a solution of the Laplace equation

$$v_{xx} + v_{yy} = 0$$

satisfying certain boundary conditions and in the course of his calculations he is led to the infinite system

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1^2 & 3^2 & 5^2 & \cdots \\ 1^4 & 3^4 & 5^4 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}. \tag{1}$$

The numbers x_k he finds by applying the finite section method are appropriate for the solution of his original problem; however, in a strict sense, they do not solve the infinite system above.

Therefore, the question arises: Under what conditions is it possible to apply the finite section method to obtain a solution of such an infinite system? The particular problem above admits the natural generalization

$$\begin{pmatrix} 1 & 1 & 1 & \cdots \\ a_0 & a_1 & a_2 & \cdots \\ a_0^2 & a_1^2 & a_2^2 & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \end{pmatrix}, \tag{2}$$

i.e., it is a system described by an *infinite Vandermonde matrix*, where we take $a_k = (2k+1)^2$ in (1). We shall examine in this paper how the finite section method works for this class.

We have to define what we mean precisely when we say that the finite section method works, as this differs from the interpretation in e.g. [1,3] where the operator is a bounded invertible operator.

We start by introducing some concepts and notations for sequence spaces; see e.g. [6]. Let ω be the vector space of all complex valued sequences, let X be a linear subspace of ω and let τ be a vector space topology on X. We assume that the set Φ of finitely supported complex sequences is contained in X ($\Phi = \{x \in \omega | \exists n_0(x) \in \mathbb{N}\ \forall n > n_0(x) : x_n = 0\}$).

We denote by $\pi_n : \omega \mapsto \mathbb{C}$ the projection onto the n'th coordinate, that is $\pi_n(x) = \pi_n(x_0, x_1, \ldots) = x_n$ and by $P_n : \omega \mapsto \omega$ the projection $P_n(x_0, x_1, x_2, \ldots, x_n, x_{n+1}, \ldots) = (x_0, x_1, \ldots, x_n, 0, 0, \ldots)$. Whenever convenient, we shall view P_n as a map from ω to $\mathbb{C}^n$.

Further, let $A(X \mapsto \omega)$ be a matrix mapping. That is,

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdots \\ a_{10} & a_{11} & a_{12} & \cdots \\ a_{20} & a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \vdots & \end{pmatrix},$$

with $a_{ij} \in \mathbb{C}$. The notation $A(X \mapsto \omega)$ used here indicates that we do not assume that A is