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The band method and inverse problems for orthogonal matrix functions of Szegő–Kreĭn type

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Dedicated to the memory of Israel Gohberg, a wonderful mathematician and a dear friend. His achievements will be a source of inspiration for many years to come

Abstract

A band method approach for solving inverse problems for certain orthogonal functions is developed. The inverse theorems for Szegő–Kreĭn matrix polynomials and for Kreĭn orthogonal entire matrix functions are obtained as corollaries of the band method results. Other examples, including a non-stationary variant of the Szegő–Kreĭn theorem, are presented to illustrate the scope of the abstract theorems.

Keywords: Band method; Inverse problems; Orthogonal polynomial; Orthogonal entire function; Non-stationary orthogonal polynomial; Matrix polynomial equation; Entire matrix function equation; Jordan chain; Root function

1. Introduction

The band method is an abstract scheme that allows one to deal with matrix-valued versions of classical interpolation problems, such as those of Schur, Carathéodory–Toeplitz and Nehari, from one point of view. The method has its origin in papers of Dym and Gohberg from the early eighties [4,5,3], and has been developed into a more final form in papers by Gohberg and co-authors in [11,12]. A comprehensive introduction to the method and additional references can be found in Chapter XXXIV of the book [7]. For more recent contributions see the article [14] and the references therein.

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In the present paper the inverse theorems for Szegő–Kreĭn matrix polynomials given in [13] are put into the context of the band method using ideas from [1]. We also use our band method results to prove various other inverse theorems, including the one for Kreĭn orthogonal entire matrix functions presented in [10].

To state our main theorem, we first recall some of the basic elements of the band method theory. Let $\mathcal{M}$ be a $*$-subalgebra of a unital C^*-algebra $\mathcal{R}$ such that the unit e of $\mathcal{R}$ belongs to $\mathcal{M}$. Assume that $\mathcal{M}$ admits a direct sum decomposition

$$\mathcal{M} = \mathcal{M}_1 \dot{+} \mathcal{M}_2^0 \dot{+} \mathcal{M}_d \dot{+} \mathcal{M}_3^0 \dot{+} \mathcal{M}_4, \tag{1.1}$$

where the summands $\mathcal{M}_1$, $\mathcal{M}_2^0$, $\mathcal{M}_d$, $\mathcal{M}_3^0$, and $\mathcal{M}_4$ are linear submanifolds of $\mathcal{M}$. The algebra $\mathcal{M}$ is called an *algebra with band structure* if, in addition, the following three conditions are satisfied

(C1) $e \in \mathcal{M}_d$,
(C2) $\mathcal{M}_1^* = \mathcal{M}_4$, $(\mathcal{M}_2^0)^* = \mathcal{M}_3^0$, and $\mathcal{M}_d = \mathcal{M}_d^*$,
(C3) the following multiplication table describes some additional restrictions on the multiplication in $\mathcal{M}$:

	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}$
$\mathcal{M}_2^0$	$\mathcal{M}_1$	$\mathcal{M}_+^0$	$\mathcal{M}_2^0$	$\mathcal{M}_c$	$\mathcal{M}_-^0$
$\mathcal{M}_d$	$\mathcal{M}_1$	$\mathcal{M}_2^0$	$\mathcal{M}_d$	$\mathcal{M}_3^0$	$\mathcal{M}_4$
$\mathcal{M}_3^0$	$\mathcal{M}_+^0$	$\mathcal{M}_c$	$\mathcal{M}_3^0$	$\mathcal{M}_-^0$	$\mathcal{M}_4$
$\mathcal{M}_4$	$\mathcal{M}$	$\mathcal{M}_-^0$	$\mathcal{M}_4$	$\mathcal{M}_4$	$\mathcal{M}_4$.

(1.2)

Here

$$\mathcal{M}_+^0 = \mathcal{M}_1 \dot{+} \mathcal{M}_2^0, \qquad \mathcal{M}_c = \mathcal{M}_2^0 \dot{+} \mathcal{M}_d \dot{+} \mathcal{M}_3^0, \qquad \mathcal{M}_-^0 = \mathcal{M}_3^0 \dot{+} \mathcal{M}_4.$$

We shall also need the linear submanifolds of $\mathcal{M}$ given by

$$\mathcal{M}_+ = \mathcal{M}_+^0 \dot{+} \mathcal{M}_d, \qquad \mathcal{M}_- = \mathcal{M}_-^0 \dot{+} \mathcal{M}_d,$$
$$\mathcal{M}_2 = \mathcal{M}_2^0 \dot{+} \mathcal{M}_d, \qquad \mathcal{M}_3 = \mathcal{M}_3^0 \dot{+} \mathcal{M}_d.$$

An element $m \in \mathcal{M}$ will be called *selfadjoint* whenever $m = m^*$.

The inverse problem we shall be dealing with in this band method setting is the following problem. Given $q \in \mathcal{M}_2$ and $a = a^* \in \mathcal{M}_d$, find $f = f^* \in \mathcal{M}_c$ such that

$$P_{\mathcal{M}_2}(fq) \in \mathcal{M}_d, \qquad P_{\mathcal{M}_d}(q^* f q) = a. \tag{1.3}$$

Here $P_{\mathcal{M}_2}$ denotes the projection of $\mathcal{M}$ onto $\mathcal{M}_2$ along the other spaces in the decomposition (1.1). In a similar way one defines other projections corresponding to subspaces in (1.1). In particular, $P_{\mathcal{M}_d}$ is the projection of $\mathcal{M}$ onto $\mathcal{M}_d$ along $\mathcal{M}_+^0 \dot{+} \mathcal{M}_-^0$, and $P_{\mathcal{M}_c}$ is the projection of $\mathcal{M}$ onto $\mathcal{M}_c$ along $\mathcal{M}_1$ and $\mathcal{M}_4$.

We shall see that under the additional condition that q has an inverse in $\mathcal{M}$ the above problem is solvable if and only if the equation

$$uq - q^* v = a \tag{1.4}$$

has a solution $u \in \mathcal{M}_2$ and $v \in \mathcal{M}_1$. The following theorem is the main result of this paper.