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indagationes mathematicae

Indagationes Mathematicae 25 (2014) 1–23

www.elsevier.com/locate/indag

Transitivity and structure of operator algebras with a metric property

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Received 2 November 2012; received in revised form 13 April 2013; accepted 15 April 2013

Communicated by M.A. Kaashoek

This paper is dedicated to the memory of Israel Gohberg, one of the giants of linear analysis and a wonderful human being

Abstract

In this paper we discuss a new metric property that some operator algebras on Hilbert space possess and some resulting consequences concerning transitivity and structure theory of such algebras.

Keywords: Transitive algebra; Invariant subspace; K-spectral set

1. Introduction

In this paper $\mathcal{H}$ will always denote a separable, infinite dimensional, complex Hilbert space, and as usual we write $\mathcal{L}(\mathcal{H})$ for the algebra of all (bounded, linear) operators on $\mathcal{H}$. We also write $\mathbf{K}$ for the ideal of compact operators in $\mathcal{L}(\mathcal{H})$ and denote the quotient (Calkin) map $\mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K}$ by π. For each $T \in \mathcal{L}(\mathcal{H})$ we employ the notation $\sigma(T)$ and $\sigma_e(T) := \sigma(\pi(T))$ for the spectrum and essential spectrum of T, respectively, and we write $\|T\|_e := \|\pi(T)\|$.

In what follows, $\mathbb{A}$ will always denote a unital, norm-closed, subalgebra of $\mathcal{L}(\mathcal{H})$ and $\mathbb{A}^{-W}$ the closure of $\mathbb{A}$ in the weak (equivalently, strong) operator topology (herein denoted WOT and SOT,

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http://dx.doi.org/10.1016/j.indag.2013.04.007

respectively). Recall that a subalgebra $\mathbb{A} \subset \mathcal{L}(\mathcal{H})$ is called *transitive* if the only subspaces left invariant by every $A \in \mathbb{A}$ are (0) and $\mathcal{H}$, and recall also that long ago, motivated by Burnside's theorem for finite dimensional spaces, R. Kadison in [9] raised the (still open) problem whether every transitive subalgebra $\mathbb{A}$ of $\mathcal{L}(\mathcal{H})$ satisfies $\mathbb{A}^{-W} = \mathcal{L}(\mathcal{H})$. The present authors, in the summer of 2005, as a consequence of their study of the construction in [1], which was itself a modification of the original constructions of Lomonosov [11,12], became interested in the following variant of the Kadison problem.

Problem 1.1. If there exists a transitive subalgebra $\mathbb{A}$ of $\mathcal{L}(\mathcal{H})$ such that $\mathbb{A}^{-W} \neq \mathcal{L}(\mathcal{H})$, is it necessarily true that $\| \ \|$ and $\| \ \|_e$ are equivalent norms on $\mathbb{A}$?

Of course, since no such transitive algebra $\mathbb{A}$ with $\mathbb{A}^{-W} \neq \mathcal{L}(\mathcal{H})$ is presently known to exist, it would certainly be difficult to give a negative answer to Problem 1.1. On the other hand, as mentioned above, the present authors, while making a detailed, in-depth, study of the construction in [1] which eventually resulted in the production of this paper, thought they saw a path to an affirmative answer to Problem 1.1. This study, over time, produced the concept of the sets $\Gamma_\alpha(y)$ defined at the beginning of Section 2 and also the concept of an algebra $\mathbb{A}$ having Property (P) defined below.

We were strongly motivated to solve Problem 1.1 affirmatively because such a result would yield immediately the existence of nontrivial invariant subspaces for a large class of operators including all operators of the form $S + K$, where S is subnormal and $K \in \mathbf{K}$ (see Section 5).

Although we have thus far failed to solve Problem 1.1 affirmatively, we have obtained some weaker results in this direction (see Theorem 6.4 and Corollary 6.5), and moreover, taking into consideration the difficulty in finding useful new consequences of positive solutions to various invariant subspace problems, we suggest that the structure theory of the class of algebras studied herein is interesting independent of the existence or non-existence of invariant subspaces. For example, our theorems show that the transitive algebras studied in [1,11,12] have the metric property (P).

As usual, we write for each $y_0 \in \mathcal{H}$ and each $\delta > 0$,

$$B(y_0, \delta) := \{y \in \mathcal{H} : \|y - y_0\| < \delta\},$$

i.e., $B(y_0, \delta)$ is the open ball in $\mathcal{H}$ centered at y_0 and having radius δ. We can now introduce the metric property referred to in the title.

Definition 1.2. With $\mathbb{A}$ and $\alpha > 0$ given and y arbitrary in $\mathcal{H} \setminus (0)$, we define

$$\Gamma_\alpha(y) := \{Ay : A \in \mathbb{A}, \ \|A\|_e \le \alpha\}, \tag{1.1}$$

and say of a (unital, norm closed) subalgebra $\mathbb{A}$ of $\mathcal{L}(\mathcal{H})$ that $\mathbb{A}$ has *Property* (P) if there exists a quadruple $(y_0, \alpha, \delta, \delta_0)$, called an *implementing quadruple*, such that

(1) $y_0 \in \mathcal{H} \setminus (0)$,
(2) $\alpha \in (0, 1/2)$,
(3) $\delta \in (0, \|y_0\|)$,
(4) $\delta_0 \in (0, (1 - 2\alpha)\delta)$,
and such that the sets $\Gamma_\alpha(y)$ have the property that
(5) for every $y \in B(y_0, \delta)^-$, $\Gamma_\alpha(y) \cap B(y_0, \delta_0)^- \neq \varnothing$, (i.e., for every y satisfying $\|y_0 - y\| \le \delta$, there exists $A_y \in \mathbb{A}$ with $\|A_y\|_e \le \alpha$ and A_y moves y into the smaller closed ball centered at y_0 with radius $\delta_0 < \delta$).