Available online at www.sciencedirect.com
ScienceDirect
Indagationes Mathematicae 25 (2014) 24–34
indagationes mathematicae
www.elsevier.com/locate/indag

Approximation and solution of a general symmetric functional equation

Hamid Rezaei*

Department of Mathematics, College of Sciences, Yasouj University, Yasouj-75914-74831, Iran

Received 23 August 2012; accepted 30 May 2013

Communicated by B. de Pagter

Abstract

In this paper, we investigate the Hyers–Ulam–Rassias stability of a general equation $f(\varphi_1(x, y)) = \varphi_2(f(x), f(y))$ in metric spaces. As a consequence, we obtain some stability results in the sense of Hyers–Ulam–Rassias.

Keywords: Functional equation; Hyers–Ulam–Rassias stability; Fixed point

1. Introduction

The stability theory of functional equations originated from the well-known Ulam problem [14], concerning the stability of homomorphisms in metric groups: Under what conditions does there exist an additive mapping near an approximately additive mapping? Hyers [5] gave a first affirmative partial answer to the question of Ulam for Banach spaces.

Theorem 1.1 (*[5]*). *Let X_1 and X_2 be Banach spaces and the function $f : X_1 \to X_2$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \le \varepsilon$$

for all $x, y \in X_1$ and some $\varepsilon > 0$. Then there exists a unique additive mapping $T : X_1 \to X_2$ such that $\|f(x) - T(x)\| \le \varepsilon$ for all $x \in X_1$.

* Tel.: +98 9177363820.
E-mail addresses: hrezaei81@gmail.com, rezaei@mail.yu.ac.ir.

0019-3577/$ - see front matter
http://dx.doi.org/10.1016/j.indag.2013.05.003

After Hyers' result many papers dedicated to this topic, extending the Ulam problem to other functional equations and generalizing Hyers' result in various directions, were published. A new direction of research in the stability theory of functional equations, called today Hyers–Ulam stability, was opened by the papers of Aoki and Rassias by considering instead of $\varepsilon > 0$ in the above theorem, an unbounded function depending on x and y [1,11]:

Theorem 1.2 (*[11]*). *Let X_1 be a normed space and X_2 a Banach space. Let $f : X_1 \to X_2$ satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta(\|x\|^p + \|y\|^p)$$

for all $x, y \in X_1$, where $\theta > 0$ and $p \in [0, 1)$. Then there exists a unique additive mapping $A : X_1 \to X_2$ such that $\|f(x) - A(x)\| \le \frac{2\theta}{2-2^p}\|x\|^p$ for all $x \in X_1$.

This exciting result of Rassias attracted a number of mathematicians who began to be stimulated to investigate the stability problems of functional equations. By regarding a large influence of S.M. Ulam, D.H. Hyers, and Th.M. Rassias on the study of stability problems of functional equations, the stability phenomenon proved by Th.M. Rassias is called the Hyers–Ulam–Rassias (HUR for short) stability.

Many mathematicians have extensively investigated the subjects on the HUR-stability. Several books cover and offer almost all classical results on the HUR-stability in an integrated and self-contained fashion [6,8,7].

If we look at the many of famous functional equation such as additive Cauchy equation, generalized additive Cauchy equations, exponential equation and logarithmic equation (see [8]), we find that they are satisfied in the following general equation:

$$f(\varphi_1(x, y)) = \varphi_2(f(x), f(y)) \tag{1.1}$$

where $\varphi_i : X_i \times X_i \to X_i$ is given function for $i = 1, 2$, $f : X_1 \to X_2$ is unknown function, X_1 is a set and X_2 is a complete metric space. The functional equation (1.1) has been appeared in [4]. The existence of solutions (1.1) and a generalization of Hyers' Theorem is investigated by Forti in [3]. He proved, under certain hypothesis on φ_1 and φ_2, that the existence of solution of the functional inequality

$$d\big(f(\varphi_1(x, y)), \varphi_2(f(x), f(y))\big) \le \beta(x, x)$$

where $\beta : X \times X \to [0, +\infty)$ is a suitable function, implies the existence of a solution of the Eq. (1.1). This means that the Eq. (1.1) has HUR-stability. For the convenience of the reader, the Forti' theorem is stated here. Our notation is slightly different from that of [3].

Let $\varphi_i : X_i \times X_i \to X_i$ is given function for $i = 1, 2$, $f : X_1 \to X_2$ is the unknown function, X_1 is a set and (X_2, d) is a complete metric space. Forti [3] proved, under certain hypotheses on φ_1, φ_2, that the existence of a solution of the functional inequality (1.1) where $\beta : X \times X \to \mathbb{R}^+$ is a suitable functions, implies the existence of a solution of the Eq. (1.1). Consider the functional equation (1.1). We put $T_i(x) = \varphi_i(x, x)$ for $i = 1, 2$ and assume that T_2 is invertible. For a nonnegative integer n, T_i^n will denote the n-th iterates of T_i and T_2^{-n} denotes the n-th iterate of T_2^{-1}. For every $x \in X_i$, we define $x^0 = x$ and $x^n = T_i^n x$ for $i = 1, 2$. For every $f : X_1 \to X_2$ and $x, y \in X_1$, we set

$$\beta(x, y) = d\big(f(\varphi_1(x, y)), \varphi_2(f(x), f(y))\big).$$