Available online at www.sciencedirect.com
ScienceDirect
Indagationes Mathematicae 25 (2014) 59–77
indagationes mathematicae
www.elsevier.com/locate/indag

Low-dimensional cohomology of Lie superalgebra $\mathfrak{sl}_{m|n}$ with coefficients in Witt or Special superalgebras

Liping Sun[a,b], Wende Liu[c,*], Boying Wu[a]

[a] *Department of Mathematics, Harbin Institute of Technology, Harbin 150006, PR China*
[b] *School of Applied Sciences, Harbin University of Science and Technology, Harbin 150080, PR China*
[c] *School of Mathematical Sciences, Harbin Normal University, Harbin 150025, PR China*

Received 31 October 2012; accepted 21 July 2013

Communicated by E.P. van den Ban

Abstract

Over a field of characteristic $p > 2$, the Witt superalgebras are viewed as modules of the special linear Lie superalgebra $\mathfrak{sl}_{m|n}$ by means of the adjoint representation. The low-dimensional cohomology groups of $\mathfrak{sl}_{m|n}$ with coefficients in Witt superalgebras are computed by virtue of the direct sum decomposition of submodules and the weight space decompositions of these submodules relative to the standard Cartan subalgebra of $\mathfrak{sl}_{m|n}$. Then we are able to obtain the low-dimensional cohomology groups of $\mathfrak{sl}_{m|n}$ with coefficients in Special superalgebras, which are $\mathfrak{sl}_{m|n}$-submodules of Witt superalgebras.

Keywords: Witt superalgebra; Special superalgebra; Cohomology

1. Introduction

The cohomology groups are of crucial importance for studying the structure and the classification of modular Lie superalgebras. However, little seems to be known about the modular Lie superalgebra cohomology when coefficient modules are not trivial or adjoint ones, except [9]. In the present article, over a field of characteristic $p > 2$, we mainly compute the low-dimensional cohomology groups of the special linear Lie superalgebra $\mathfrak{sl}_{m|n}$ with coefficients in the restricted Witt superalgebra W and Special superalgebra S viewed as $\mathfrak{sl}_{m|n}$-modules in the natural fashion. The Witt superalgebra W and Special superalgebra S (see [8]) are

* Corresponding author.
E-mail addresses: wendeliu@ustc.edu.cn, hhslp@126.com (W. Liu).

0019-3577/$ - see front matter
http://dx.doi.org/10.1016/j.indag.2013.07.006

analogous to the finite-dimensional modular Lie algebras of Cartan type [5] or the infinite-dimensional Lie superalgebras of Cartan type defined by even differential forms over a field of characteristic zero [2]. In general, the classical Lie(super)algebra, which contained in the null of a graded simple Lie(super)algebra, plays an important role in characterizing the structure of the Lie(super)algebra under consideration.

In this paper, we describe the structure of $\mathfrak{sl}_{m|n}$-module W and compute the weight space decompositions of certain submodules of W relative to the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}_{m|n}$. Then the work under consideration is reduced to computing the cohomology groups with coefficients in certain submodules and computing the so-called weight-derivations of $\mathfrak{sl}_{m|n}$ to these submodules, that is, the derivations preserving the $\mathfrak{h}^*$-gradings. Since the Special superalgebra S contains $\mathfrak{sl}_{m|n}$ as a subalgebra and W contains S as a $\mathfrak{sl}_{m|n}$-submodule, as an application, we then use the results we have obtained for W to compute the low-dimensional cohomology of $\mathfrak{sl}_{m|n}$ with coefficients in S.

It should be worthwhile to talk about the following features of this paper. Firstly, instead of using a direct computation as used in [3,4], we adopt two methods to simplify the computations as mentioned above. Secondly, the results and methods used in this paper do not work for the low-dimensional cohomology of $\mathfrak{sl}_{2|1}$ with coefficients in W and S, in which case this has been treated specially by another way (see [7]). At last, our results show that the cohomology groups for Lie superalgebras over a field of finite characteristic may be different sharply from the ones for Lie superalgebras over a field of characteristic zero [6] (see Remark 5.8).

Throughout the paper, all the vector spaces are over a field $\mathbb{F}$ of characteristic $p > 2$ and are finite-dimensional. Let $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ be the two-element field. The symbol $|x|$ or $\mathrm{zd}(x)$ denotes the $\mathbb{Z}_2$-degree or $\mathbb{Z}$-degree of a $\mathbb{Z}_2$-homogeneous or $\mathbb{Z}$-homogeneous element x respectively. Write $\langle v_1, \dots, v_k\rangle$ for the subspace spanned by $v_1, \dots, v_k$ over $\mathbb{F}$.

Fix two positive integers $\bar{m}, \bar{n} > 1$. For convenience, put

$$\mathbf{Y}_0 = \{1, 2, \dots, \bar{m}\}, \qquad \mathbf{Y}_1 = \{\bar{m}+1, \dots, \bar{m}+\bar{n}\}, \qquad \mathbf{Y} = \mathbf{Y}_0 \cup \mathbf{Y}_1.$$

Let $\mathcal{O}(\bar{m})$ be the divided power algebra spanned by the standard basis

$$\{x^{(\alpha)} := x_1^{\alpha_1} \cdots x_{\bar{m}}^{\alpha_{\bar{m}}} \mid \alpha = (\alpha_1, \dots, \alpha_{\bar{m}}) \in \mathbb{Z}^{\bar{m}},\ 0 \le \alpha_i \le p-1\}$$

with multiplication $x^{(\alpha)}x^{(\beta)} = \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}$, where $\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^{\bar{m}} \binom{\alpha_i+\beta_i}{\alpha_i}$. Denote by $\Lambda(\bar{n})$ the exterior algebra of $\bar{n}$ variables $x_{\bar{m}+1}, \dots, x_{\bar{m}+\bar{n}}$, which has a standard basis

$$\{x^u := x_{\bar{m}+1}^{\alpha_{\bar{m}+1}} \cdots x_{\bar{m}+\bar{n}}^{\alpha_{\bar{m}+\bar{n}}} \mid u = (\alpha_{\bar{m}+1}, \dots, \alpha_{\bar{m}+\bar{n}}) \in \mathbb{Z}^{\bar{n}},\ \alpha_i = 1 \text{ or } 0\}.$$

The tensor product $\mathcal{O}(\bar{m}, \bar{n}) := \mathcal{O}(\bar{m}) \otimes \Lambda(\bar{n})$ is an associative superalgebra. We write $x^{(\alpha)} \otimes x^u$ as $x^{(\alpha)}x^u$ in $\mathcal{O}(\bar{m}, \bar{n})$. Let $\partial_1, \dots, \partial_{\bar{m}+\bar{n}}$ be the special superderivations of $\mathcal{O}(\bar{m}, \bar{n})$, such that $\partial_i(x^{(\alpha)}) = x^{(\alpha-\varepsilon_i)}$ for $i \in \mathbf{Y}_0$ and $\partial_i(x_j) = \delta_{i,j}$ for $i, j \in \mathbf{Y}_1$. The finite-dimensional *Witt superalgebra*, denoted by $W(\bar{m}, \bar{n})$, is a Lie superalgebra spanned by the standard basis $\{f\partial_i \mid f \in \mathcal{O}(\bar{m}, \bar{n}),\ i \in \mathbf{Y}\}$. Note that $W(\bar{m}, \bar{n})$ has a standard $\mathbb{Z}$-gradation induced by $\mathrm{zd}(x_i) = 1$ and $\mathrm{zd}(\partial_i) = -1$ for $i \in \mathbf{Y}$. Let div be the divergence:

$$\mathrm{div} : W(\bar{m}, \bar{n}) \longrightarrow \mathcal{O}(\bar{m}, \bar{n}), \qquad f\partial_i \longmapsto (-1)^{|f||\partial_i|}\partial_i(f).$$

Put $\bar{S}(\bar{m}, \bar{n}) = \langle D \in W(\bar{m}, \bar{n}) \mid \mathrm{div}\,(D) = 0\rangle$. Then $\bar{S}(\bar{m}, \bar{n})$ is a subalgebra of $W(\bar{m}, \bar{n})$. Its derived algebra $S(\bar{m}, \bar{n}) := [\bar{S}(\bar{m}, \bar{n}), \bar{S}(\bar{m}, \bar{n})]$ is a simple Lie superalgebra, called the *Special superalgebra*. For convenience, we simply write W, S and $\mathcal{O}$ for $W(\bar{m}, \bar{n})$, $S(\bar{m}, \bar{n})$ and $\mathcal{O}(\bar{m}, \bar{n})$, respectively.