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Extreme contractive operators on Stone f-algebras

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Abstract

An Archimedean semiprime f-algebra A for which

$$I_A \wedge f \in A \quad \text{for all } f \in A$$

is called a Stone f-algebra, where I_A is the identity operator on A. Moreover, an operator T between two Stone f-algebras A and B is said to be contractive if

$$f \in A \quad \text{and} \quad 0 \leq f \leq I_A \quad \text{imply } 0 \leq Tf \leq I_B.$$

The set $\mathcal{K}(A, B)$ of all positive contractive operators from A into B is a convex set. This paper characterizes extreme points in $\mathcal{K}(A, B)$. In this regard, we prove that $T \in \mathcal{K}(A, B)$ is extreme if and only if T is an algebra homomorphism. Furthermore, we show that $T \in \mathcal{K}(A, B)$ is extreme if and only if T is a Stone operator, meaning that,

$$T(I_A \wedge f) = I_B \wedge Tf \quad \text{for all } f \in A.$$

Keywords: Algebra homomorphism; Contractive operator; Convex set; Extreme point; Positive operator; Riesz homomorphism; Stone f-algebra; Stone operator

1. Introduction

One of the major problems in Operator Theory is to describe extreme points in an appropriate convex set of (linear) operators from an algebra to another. The importance of this theme has

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steadily grown in the course of time, in part because it is usually developed around extension problems. In this paper we are concerned in the extreme points in the convex set of all positive contractive operators on Archimedean semiprime f-algebras. A short historical account seems in order.

Let A and B be Archimedean f-algebras with unit elements I_A and I_B, respectively. A positive operator $T : A \to B$ is called a *Markov operator* if T is *identity preserving*, that is, $TI_A = I_B$. The set $\mathcal{M}(A, B)$ of all Markov operators is a convex set. Relaying heavily on an idea by van Putten [8], Huijsmans and de Pagter [5] remarkably proved that if $T \in \mathcal{M}(A, B)$, then the following are equivalent.

(i) T is an extreme point in $\mathcal{M}(A, B)$.
(ii) T is an algebra homomorphism.
(iii) T is a Riesz homomorphism.

The main purpose of our work is to obtain the corresponding characterizations of extreme points in the convex set of all contractive positive operators on what we will call Stone f-algebras (see [4] for the case of algebras of real-valued continuous functions). To be more precise, let A be an arbitrary Archimedean f-algebra. It is well-known that if A is semiprime (i.e., with no nonzero nilpotent elements) then A is (isomorphic with) a sub f-algebra of the Archimedean f-algebra Orth(A) of all orthomorphisms on A (see, e.g., [3, Section 12.3]). By the way, $A = \text{Orth}(A)$ if and only if A has a unit element. Notice here that the identity operator I_A on A is the unit element of Orth(A). We call the Archimedean semiprime f-algebra A a *Stone f-algebra* if

$$I_A \wedge f \in A \quad \text{for all } f \in A.$$

So, let A and B be Stone f-algebras. An operator $T : A \to B$ is said to be *contractive* if

$$0 \le Tf \le I_B \quad \text{for all } f \in A \text{ with } 0 \le f \le I_A.$$

Obviously, the set $\mathcal{K}(A, B)$ of all contractive positive operators from A into B is a convex set. Our first result asserts that an operator $T \in \mathcal{K}(A, B)$ is an extreme point in $\mathcal{K}(A, B)$ if and only if T is an algebra homomorphism. If truth be told, this characterization of extreme point in $\mathcal{K}(A, B)$ is rather expected, although its proof is far from being trivial. But what about the other characterization, viz., do extreme points in $\mathcal{K}(A, B)$ and Riesz homomorphisms in $\mathcal{K}(A, B)$ coincide? Of course, this is not true since the operator $T : \mathbf{R} \to \mathbf{R}$ defined by $Tf = f/2$ for all $f \in \mathbf{R}$ is a contractive Riesz homomorphism but fails to be an algebra homomorphism and then to be an extreme point (the symbol $\mathbf{R}$ is used here to indicate the reals). The right question seems therefore to be the following. What could be a 'lattice' characterization of extreme points in $\mathcal{K}(A, B)$ that would replace the third condition in the aforementioned result of Huijsmans and de Pagter? We were able to find a quite surprising answer to this question. Indeed, call an operator $T : A \to B$ a *Stone operator* if

$$T(I_A \wedge f) = I_B \wedge Tf \quad \text{for all } f \in A.$$

As we shall see next, it turns out that Stone operators from A into B coincide with extreme points in $\mathcal{K}(A, B)$, which we believe to be a quite satisfactory solution of our problem.

Finally, we point out that we will use the classical monographs [7,9] as unique sources of unexplained terminology and notations. Nevertheless, it would be helpful to keep the fundamental papers [6,5] handy.