Available online at www.sciencedirect.com
ScienceDirect
Indagationes Mathematicae 25 (2014) 145–161
indagationes mathematicae
www.elsevier.com/locate/indag

Riemann–Liouville abstract fractional Cauchy problem with damping☆

Zhan-Dong Mei*, Ji-Gen Peng

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, China

Received 23 October 2012; received in revised form 4 October 2013; accepted 9 October 2013

Communicated by J. Wiegerinck

Abstract

This paper is concerned with Riemann–Liouville abstract fractional Cauchy Problems with damping. The notion of Riemann–Liouville fractional (α, β, c) resolvent is developed, where $0 < \beta < \alpha \leq 1$. Some of its properties are obtained. By combining such properties with the properties of general Mittag-Leffler functions, existence and uniqueness results of the strong solution of Riemann–Liouville abstract fractional Cauchy Problems with damping are established. As an application, a fractional diffusion equation with damping is presented.

Keywords: Abstract fractional Cauchy problems; Riemann–Liouville fractional (α, β, c) resolvent; Strong solution

1. Introduction

In this paper, we are concerned with the following Riemann–Liouville abstract fractional Cauchy problem with damping

$$(\text{RLAFCP})_x \begin{cases} D_t^{\alpha} u(t) + cD_t^{\beta} u(t) = Au(t), \quad t > 0, \\ (g_{1-\alpha} * u)(0) = x, \end{cases}$$

where $0 < \beta < \alpha \leq 1, u(\cdot)$ is the state, $A : D(A) \subset X \to X$ is a closed linear operator, $(X, \|\cdot\|)$ is a Banach space, $D(A)$ is the domain of A endowed with the graph norm $\|\cdot\|_{D(A)} =$

☆ This work was supported by the National Science Foundation of China under contract nos. 11301412 and 11131006 and partly by the Fundamental Research Funds for the Central Universities under the contact no. 2012jdhz52.
* Corresponding author. Tel.: +86 13991224735.
E-mail addresses: zhdmei@mail.xjtu.edu.cn (Z.-D. Mei), jgpeng@mail.xjtu.edu.cn (J.-G. Peng).

0019-3577/$ - see front matter
http://dx.doi.org/10.1016/j.indag.2013.10.001

$\|\cdot\| + \|A\cdot\|$, D_t^α and D_t^β are respectively the α-order and β-order Riemann–Liouville fractional derivative operators, $g_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, c is a real number. If the coefficient $c = 0$, we call $(\text{RLAFCP})_x$ to be Riemann–Liouville abstract fractional Cauchy problem. For fractional calculus involving fractional derivatives, we refer to [12].

Fractional derivatives possess memorizing properties, which makes fractional derivative more suitable than derivative of integer-order to describe the properties of various real materials. For example, fractional Cauchy problems have been used to model efficiently anomalous diffusion on fractals (physical objects of fractional dimension, like some amorphous semiconductors or strongly porous materials; see [1,22] and the references therein), fractional random walk, etc. Berens and Westphal [4] first considered the abstract Cauchy problem involving the Riemann–Liouville fractional derivative operator. Hilfer studied in [15] the Riemann–Liouville fractional diffusion equation of order $0 < \alpha < 1$

$$D_t^\alpha f(r,t) = C_\alpha \Delta f(r,t)$$

where $f(r,t)$ denotes the unknown field and C_α denotes the fractional diffusion constant with dimensions $[cm/s^\alpha]$; he pointed out that the initial condition should be given in the form of $g_{1-\alpha}(t) * f(r,t)|_{t\to 0^+}$. This implies that the initial condition should have memorizing property. Heymans and Podlubny [14] demonstrated that in some material, the initial conditions for fractional differential equations with Riemann–Liouville derivatives have strong physical meaning, and that the corresponding quantities can be obtained from measurements.

Motivated by the fact that semigroup theory is closely related to the first-order abstract Cauchy problem, Da Prato and Iannelli [9] introduced the concept of resolvent families. Oka [25] introduced integrated solution families. Lizama [20] introduced (a,k)-regularized resolvents with a being continuous on $[0,\infty)$. Bazhlekova [3] carried out research on fractional evolution equations by using solution operators. Peng and Li [26] developed the notion of fractional semigroup to study fractional differential equations. Obviously, the above "families" have a common feature, that is, the families are strongly continuous at zero. However, Riemann–Liouville abstract fractional Cauchy problems with order $0 < \alpha < 1$ possess a singularity at zero. The singularity makes the above "families" unsuitable to study such Riemann–Liouville abstract fractional Cauchy problems.

In order to study the properties of Riemann–Liouville abstract fractional Cauchy problem, Li and Peng [19] recently developed a new notion, namely α-order fractional resolvent. Concretely, they considered a Riemann–Liouville abstract fractional Cauchy problem with

$$\lim_{t\to 0^+} \Gamma(\alpha) t^{1-\alpha} u(t) = x \tag{1}$$

instead of the initial value condition $(g_{1-\alpha} * u)(0) = x$ and defined the notion α-order fractional resolvent by

Definition 1 (*[19]*)**.** Let $0 < \alpha < 1$. A family $\{T(t)\}_{t>0}$ of bounded linear operators on Banach space X is called an α-order fractional resolvent if it satisfies the following assumptions:

(P1) for any $x \in X$, $T(\cdot)x \in C((0,\infty), X)$, and

$$\lim_{t\to 0+} \Gamma(\alpha) t^{1-\alpha} T(t)x = x \quad \text{for all } x \in X; \tag{2}$$

(P2) $T(s)T(t) = T(t)T(s)$ for all $t, s > 0$;