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Review

Characterization and stability of the essential spectrum based on measures of polynomially non-strict singularity operators

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Abstract

In this study, we consider measures of polynomially non-strict singularity operators, which play crucial roles in the characterization and the stability of the Schechter essential spectrum of unbounded operators.

Keywords: Fredholm operator; Measure of non-strict singularity; Polynomial of strict singular operator; Schechter essential spectrum

Contents

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1. Introduction

In recent years, one of the most interesting problems addressed in studies of the essential spectra of linear operators on Banach spaces is the invariance of the essential spectra under (additive) perturbations that belong to any two-sided ideal of the algebra of bounded linear operators. This problem has generated a vast body of important results in the spirit of [6,7,15]. Motivated by the notion of the measure of non-strict singularity, g, introduced by Schechter in [18] (which has proved to be a useful tool in areas such as operator theory and functional analysis) and the theory of polynomially strict singular operators, we focus on a new characterization of the Schechter essential spectrum. In particular, we aim to determine the conditions that must be imposed on closed densely defined linear operators T and S in order to investigate the characterization of the essential spectrum in a good manner. Specifically, we define the set of polynomially strict singular operators, $\mathcal{P}_S$, as:

$$\mathcal{P}_S = \Bigg\{ A \in \mathcal{L}(X), \text{ such that a nonzero complex polynomial} \\ P(z) := \sum_{k=0}^{p} a_k z^k, \text{ exists that satisfies } P\left(\frac{1}{n}\right) \neq 0, \ \forall n \in \mathbb{Z}^* \text{ and } P(A) \in \mathcal{S}(X) \Bigg\}$$

and the Schechter essential spectrum is characterized by

$$\sigma_{ess}(T) = \bigcap_{S \in \mathcal{S}_T \cup \mathcal{G}_T} \sigma(T+S),$$

where

$$\mathcal{S}_T = \{S \in \mathcal{C}(X), S \text{ is } T\text{-bounded and } S(\lambda - T - S)^{-1} \in \mathcal{P}_S, \ \forall \lambda \in \rho(T+S)\}$$

and

$$\mathcal{G}_T = \{S \in \mathcal{C}(X), S \text{ is } T\text{-bounded and } \text{ a nonzero complex polynomial } P \\ \text{exists that satisfies } P(-1) \neq 0 \text{ and } |P| g(S(\lambda - T - S)^{-1}) < |P(-1)|, \\ \forall \lambda \in \rho(T+S)\}.$$

One of the central questions in the study of the stability of the Schechter essential spectrum is determining the practical criteria that we must impose on T and S in order to extend previous results, e.g., [1,2,6], to a large class of perturbations that contains the ideal of strict singular operators, which we show in Theorem 4.1 (see Section 4) under the following hypotheses:

(i) for $T \in \mathcal{C}(X)$ that satisfies the hypothesis $(\mathcal{H})$ (see Section 2),
(ii) $S \in \mathcal{S}_T$ such that for all $p \in \mathbb{Z}$, $\rho_{ess}(T + pS)$ is a connected set of $\mathbb{C}$,

and such that $\sigma_{ess}(T) = \sigma_{ess}(T+S)$.

In order to state our results, we need to fix some notations and assumptions that we employ in this study. Let $(X, \|.\|)$ be an infinite-dimensional Banach space and let $\mathcal{C}(X)$ be the set of all closed densely defined linear operators on X. We denote $\mathcal{L}(X)$ (resp. $\mathcal{K}(X)$) as the space of all bounded linear operators (resp. the subspace of all compact operators) on X. For $A \in \mathcal{C}(X)$, we use $\mathcal{N}(X) \subseteq X$ (resp. $\mathcal{R}(X) \subseteq X$) to denote the null space (resp. the range) of A. We define the nullity, $\alpha(A)$, (resp. the deficiency, $\beta(A)$) as the dimension of $\mathcal{N}(A)$ (resp. the codimension of