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# On a combinatorial sum

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**Abstract**

Let

$$P_n(x) = (1-x)^{n+1} \sum_{k=0}^{n} \binom{n+k}{k} x^k.$$

We determine all real numbers $x$ such that the sequence $\{P_n(x)\}_{n=0}^{\infty}$ is concave, convex, and completely monotonic, respectively.


*Keywords:* Combinatorial sum; Concave; Convex; Completely monotonic; Psi function

## 1. Introduction

We define the polynomial $P_{m,n}$ by

$$P_{m,n}(x) = (1-x)^{n+1} \sum_{k=0}^{m} \binom{n+k}{k} x^k, \tag{1.1}$$

where $m$ and $n$ are non-negative integers. The elegant formula

$$P_{m,n}(x) + P_{n,m}(1-x) = 1 \tag{1.2}$$

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which is known as identity of Chaundy and Bullard, was published in 1960; see [5]. A survey of different proofs of (1.2) as well as interesting historical comments on this subject can be found in [8] and [9]. Among others, the authors point out in [9] that identity (1.2) already was given implicitly in 1713 by Mountmort.

Here, we study (1.1) for the special case $m = n$ and set

$$P_n(x) = P_{n,n}(x) = \sum_{k=0}^{2n+1} c_{k,n} x^k$$

with

$$c_{k,n} = \sum_{\substack{\mu+\nu=k \\ 0\le\mu\le n+1;\, 0\le\nu\le n}} (-1)^{\mu} \binom{n+1}{\mu}\binom{n+\nu}{n}.$$

The work on this paper has been inspired by a recently published article by Aharonov and Elias [2], who use a linear differential equation of the first order to establish the representation

$$P_n(x) = \frac{\int_x^1 t^n (1-t)^n dt}{B(n+1, n+1)}, \tag{1.3}$$

where

$$B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$$

denotes Euler's beta function. Aharonov and Elias apply (1.3) to obtain a new proof for the well-known identity

$$\sum_{k=0}^{n} \binom{n+k}{n} 2^{-k} = 2^n. \tag{1.4}$$

In fact, putting $x = 1/2$ in (1.3) leads to (1.4).

In what follows we study further properties of $P_n(x)$. More precisely, we answer the following questions.

(I) For which real numbers $x$ is the sequence $\{P_n(x)\}_{n=0}^{\infty}$ concave and for which $x$ is this sequence convex?

(II) For which real numbers $x$ is the sequence $\{P_n(x)\}_{n=0}^{\infty}$ completely monotonic?

We recall that a sequence $\{a_n\}_{n=0}^{\infty}$ is said to be completely monotonic if

$$(-1)^k \Delta^k a_n \ge 0 \quad \text{for } n, k = 0, 1, 2, \ldots,$$

where

$$\Delta^0 a_n = a_n, \qquad \Delta^k a_n = \Delta^{k-1} a_{n+1} - \Delta^{k-1} a_n \quad (n = 0, 1, 2, \ldots;\ k = 1, 2, \ldots).$$

A result of Hausdorff states that $\{a_n\}_{n=0}^{\infty}$ is completely monotonic if and only if there exists a bounded non-decreasing function $b(t)$, $0 \le t \le 1$, such that

$$a_n = \int_0^1 t^n db(t) \quad \text{for } n = 0, 1, 2, \ldots;$$

see [13, p. 108].