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Indagationes Mathematicae 25 (2014) 872–900

indagationes mathematicae

www.elsevier.com/locate/indag

Poisson varieties from Riemann surfaces☆

Philip Boalch

Institut des Hautes Études Scientifiques et CNRS, Le Bois-Marie, 35 route de Chartres, 91440 Bures-sur-Yvette, France

Abstract

Short survey based on talk at the Poisson 2012 conference. The main aim is to describe and give some examples of wild character varieties (naturally generalising the character varieties of Riemann surfaces by allowing more complicated behaviour at the boundary), their Poisson/symplectic structures (generalising both the Atiyah–Bott approach and the quasi-Hamiltonian approach), and the wild mapping class groups.

1. Introduction

To fix ideas recall that there is already a much-studied class of Poisson varieties attached to Riemann surfaces, as follows.

Let G be a connected complex reductive group, such as $G = \mathrm{GL}_n(\mathbb{C})$, with a nondegenerate invariant symmetric bilinear form on its Lie algebra. Then there is a Poisson variety, the G-character variety, attached to any Riemann surface Σ:

$$\Sigma \mapsto \mathrm{Hom}(\pi_1(\Sigma), G)/G, \tag{1}$$

taking the space of conjugacy classes of representations in G of the fundamental group. In brief these spaces are complex symplectic if Σ is compact (without boundary) and in general they are Poisson with symplectic leaves obtained by fixing the conjugacy class of the monodromy around each boundary component.

The first symplectic approach to such spaces of fundamental group representations was analytic [7] and subsequently there were many alternative, more algebraic, approaches to these character varieties, such as [46,57,43,52,54,77,5,47,4]. Several applications are discussed in the surveys [6,8].

☆ Research supported by grants ANR-13-BS01-0001-01 (VarGen), ANR-13-IS01-0001-01 (SISYPH), 09-JCJC-0102-01 (RepRed).

E-mail address: boalch@ihes.fr.

http://dx.doi.org/10.1016/j.indag.2014.07.004

One of the interesting features is that these character varieties are finite dimensional Poisson manifolds which, unlike the vast majority of the Poisson manifolds appearing in classical mechanics, cannot be constructed out of finite dimensional cotangent bundles or coadjoint orbits. Rather, they are finite dimensional spaces naturally obtained from infinite dimensions, and this is the approach taken by Atiyah–Bott [7] (and this fact provided motivation for the above listed authors to seek a purely finite-dimensional approach).

However if one reads a little further in the Poisson literature it soon becomes apparent that there are other important Poisson manifolds which do not fit into this framework (of either being constructed out of finite dimensional cotangent bundles/coadjoint orbits or from moduli spaces of flat connections).

For example the Drinfeld–Jimbo quantum group $U_q(\mathfrak{g})$ has an integral structure in which we can put $q = 1$ and obtain the following Poisson manifold:

$$G^* := \{(b_+, b_-) \in B_+ \times B_- \mid \delta(b_+)\delta(b_-) = 1\} \tag{2}$$

where $B_\pm \subset G$ are a pair of opposite Borel subgroups and $\delta : B_\pm \to T$ is the natural projection onto the maximal torus $T = B_+ \cap B_-$. (Thus if $G = \mathrm{GL}_n(\mathbb{C})$ we could take T to be the diagonal subgroup and $B_\pm$ to be the upper/lower triangular subgroups and then δ is the map taking the diagonal part.) This statement is proved in [34,35] Theorem p. 86 Section 12.1, and an earlier version of this result at the level of formal groups is due to Drinfeld, cf. [40, Section 3]. These Poisson manifolds are also discussed in [73,42,3,62].

Two facts suggest that there might be some link between G^* and spaces of connections:

(1) G^* is constructed in terms of the algebraic groups $T, B_\pm$ (rather than say their Lie algebras) and algebraic groups were first considered, autonomously from Lie theory, as "Galois groups" for differential equations (see e.g. Chevalley's MR reviews of [59–61]), so naively one might guess that any natural space involving algebraic groups will have a 'differential equations' counterpart,[1]

(2) The symplectic leaves of G^* are obtained by fixing the conjugacy class of the product $b_-^{-1}b_+ \in G$ (which is reminiscent of fixing the conjugacy class of monodromy around the boundary above).

Such considerations lead to the following:

Question. Is there a class of Poisson varieties associated to connections on Riemann surfaces, generalising the above spaces of fundamental group representations, and which includes the spaces G^*?

The main aim of this note is to describe the answer to this question (from [15,17,16,18,19]), and some more recent developments [24,30]. In brief the answer is to consider more general algebraic connections than those parameterised by representations of the fundamental group—this algebraic viewpoint means that such moduli spaces are often overlooked in differential geometry, even though they parameterise objects well-known in algebraic geometry and the theory of differential equations, as will be explained. It turns out that spaces of such (more general) connections also have natural Poisson/symplectic structures. The classical ideas of Riemann and others on Fuchsian differential equations and the behaviour of their solutions (monodromy) lead to the character varieties, whereas more recent ideas, understanding the behaviour of solutions of non-Fuchsian equations, lead to the wild character varieties we will describe here.

[1] Another instance of this philosophy is the fact that the Grothendieck simultaneous resolution is a moduli space of ("logahoric") connections [26,27].