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A survey on stability and rigidity results for Lie algebras[☆]

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Abstract

We give simple and unified proofs of the known stability and rigidity results for Lie algebras, Lie subalgebras and Lie algebra homomorphisms. Moreover, we investigate when a Lie algebra homomorphism is stable under all automorphisms of the codomain (including outer automorphisms).

Keywords: Deformations; Rigidity; Lie algebras; Implicit function theorems; Kuranishi space

1. Introduction

In this paper we address the following well known, classical problems about the stability/rigidity of Lie algebras:

Problem 1 (*Rigidity of Lie Algebras*)**.** Given a Lie bracket μ on a vector space $\mathfrak{g}$, when is it true that every Lie bracket μ' sufficiently close to μ is of the form $\mu' = A \cdot \mu$ for some $A \in \mathrm{GL}(\mathfrak{g})$ close to the identity?

In the problem above, $(A \cdot \mu)(u, v) := A\mu(A^{-1}u, A^{-1}v)$. A Lie algebra which satisfies the condition above will be called **rigid**.

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Problem 2 (*Rigidity of Lie Algebra Homomorphisms*)**.** Given a Lie algebra homomorphism $\rho : \mathfrak{h} \to \mathfrak{g}$, when is it true that every Lie algebra homomorphism $\rho' : \mathfrak{h} \to \mathfrak{g}$ sufficiently close to ρ is of the form $\rho' = \mathrm{Ad}_g \circ \rho$ for some $g \in G$ close to the identity?

A Lie algebra homomorphism satisfying the condition above will be called a **rigid homomorphism**.

Problem 3 (*Rigidity of Lie Subalgebras*)**.** Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, when is it true that every subalgebra $\mathfrak{h}' \subset \mathfrak{g}$ sufficiently close to $\mathfrak{h}$ is of the form $\mathfrak{h}' = \mathrm{Ad}_g(\mathfrak{h})$ for some $g \in G$ close to the identity?

A subalgebra satisfying the condition above will be called a **rigid subalgebra**.

Problem 4 (*Stability of Lie Algebra Homomorphisms*)**.** Given a Lie algebra homomorphism $\rho : \mathfrak{h} \to \mathfrak{g}$, when is it true that for every Lie algebra $\mathfrak{g}'$ sufficiently close to $\mathfrak{g}$, there exists a homomorphism $\rho' : \mathfrak{h} \to \mathfrak{g}'$ close to ρ?

A homomorphism satisfying the condition above will be called **stable**.

Problem 5 (*Stability of Lie Subalgebras*)**.** Given a Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, when is it true that for every Lie algebra $\mathfrak{g}'$ sufficiently close to $\mathfrak{g}$, there exists a Lie subalgebra $\mathfrak{h}' \subset \mathfrak{g}'$ close to $\mathfrak{h}$?

A subalgebra satisfying the condition above will be called a **stable subalgebra**.

As a consequence of the formalism used to address these stability problems, we will give an answer also to the following problem:

Problem 6. Given a Lie algebra $\mathfrak{g}$, when is a neighborhood of $\mathfrak{g}$ smooth in the space of all Lie algebra structures?

Answers to these problems are given in [5–9]. Our solutions rely on the following analytic tools:

(1) A version of the implicit function theorem (Proposition 4.3), which guarantees that an orbit of a group action is locally open.
(2) A stability result for the zeros of a vector bundle section (Proposition 4.4).
(3) Kuranishi's description of zero sets.

Our approach – which has been inspired by [3,2] in the rigidity results, and by [1] in the stability results – allows us to answer Problems 1–6 in a simple and unified manner. Moreover the solution of an extension of Problem 2 (Theorem 5.10) is not present in these papers. In forthcoming work of the authors, infinite-dimensional versions of tools 1. and 2. – which are Propositions 4.3 and 4.4, respectively – will be used to prove stability/rigidity results in the context of Lie algebroids.

As a preparation for answering the problems mentioned above, we first review the infinitesimal deformation theory of Lie algebras, Lie subalgebras, and Lie algebra homomorphisms, respectively.

2. Lie algebra cohomology

Infinitesimally, rigidity and stability problems translate into linear algebra problems, which take place in chain complexes associated to Lie algebras and their homomorphisms. We briefly recall the construction of these complexes. For more details, see [4] and references therein.

Given a Lie algebra $\mathfrak{g}$ and a representation $r : \mathfrak{g} \to \mathfrak{gl}(V)$, one obtains a chain complex called the **Chevalley–Eilenberg complex of $\mathfrak{g}$ with coefficients in V** as follows: the cochains