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On the displacement function of isometries of Euclidean buildings

Carlos Ramos-Cuevas

Mathematisches Institut, Universität München, Theresienstr. 39, D-80333 München, Germany

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Abstract

In this note we study the displacement function $d_g(x) := d(gx, x)$ of an isometry g of a Euclidean building. We give a lower bound for $d_g(x)$ depending on the distance from x to the minimal set of g. This answers a question of Rousseau (2001, 4.8) and Rapoport–Zink (1999, 2.2).

Keywords: Euclidean buildings; Displacement function; Semisimple isometries; CAT(0) spaces

1. Introduction

Let (X, d) be a metric space and let $g \in Isom(X)$ be an isometry of X. The *displacement function* $d_g : X \to [0, \infty)$ of g is the function given by $d_g(x) := d(gx, x)$. The infimum $\delta_g := \inf\{d_g(x) \mid x \in X\}$ of this function is called the *displacement* or *translation length* of g. Let us denote with $\mathrm{Min}_g := \{x \in X \mid d_g(x) = \delta_g\}$ the subset of X where this infimum is attained. We call Min_g the *minimal set* of g. We can divide the isometries of X in three classes depending on the behavior of their displacement functions: we say that g is *elliptic* if it fixes a point, i.e. $\mathrm{Min}_g \neq \emptyset$ and $\delta_g = 0$; *hyperbolic* or *axial* if $\mathrm{Min}_g \neq \emptyset$ and $\delta_g > 0$; and *parabolic* if $\mathrm{Min}_g = \emptyset$. We say that g is *semisimple* if it is elliptic or hyperbolic.

Suppose now that (X, d) is a complete CAT(0) space. Let $C \subset X$ be a closed convex subset. Then the *nearest point projection* $p = p_C : X \to C$ is a well defined 1-Lipschitz map. It

E-mail address: cramos@mathematik.uni-muenchen.de.

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holds that the angle $\angle_{p(x)}(x, z)$ at $p(x)$ between x and z is $\geq \frac{\pi}{2}$ for all $z \in C$ and if $y \in X$ lies on the segment $xp(x)$ between $x \in X$ and its projection $p(x)$, then $p(y) = p(x)$ (see e.g. [1, Prop. II.2.4]). Let $g \in Isom(X)$ be a semisimple isometry, then the convexity of the distance function implies that the minimal set Min_g is a closed convex subset and the displacement function is an increasing function of the distance to the minimal set. More precisely, let $p : X \to \mathrm{Min}_g$ be the nearest point projection. Suppose $\rho : [0, \infty) \to X$ is a geodesic ray with $\rho(0) \in \mathrm{Min}_g$ and $p(\rho(t)) = \rho(0)$ for all t, that is, ρ is orthogonal to Min_g. It follows from the convexity of the distance function and the 1-Lipschitz property of the projection that $d_g \circ \rho$ is a strictly monotonically increasing function. We are interested in understanding the growth of this function in the case when X is a Euclidean building. An upper bound for $d_g \circ \rho$ given by $\delta_g + 2t$ follows easily in general from the triangle inequality. It is also not difficult to give examples where this bound is attained. We are therefore interested in giving lower bounds for the function $d_g \circ \rho$.

Euclidean buildings are a special kind of CAT(0) spaces. They are built up from top dimensional building blocks called *apartments* isometric to a Euclidean space. These apartments are glued together following a pattern given by a Euclidean Coxeter complex (a class of groups of isometries of the Euclidean space generated by reflections). This gluing pattern and a certain angle rigidity in the space of directions at points force the geometry of Euclidean buildings to have some discreteness nature. It follows that in many cases a geometric property at a point or a segment does not depend on the point or segment in question or even on the Euclidean building itself but only on the type of its Coxeter complex and the relative position of the point or segment with respect to this Coxeter complex.

If $X \cong \mathbb{R}^k$ is a Euclidean space, then the function $d_g \circ \rho$ above is given by $d_g(\rho(t)) = \sqrt{\delta_g^2 + Ct^2}$ where $C > 0$ is a constant depending only on the linear part of g. It is reasonable to expect a similar behavior of this function in the case of Euclidean buildings. We show the following (this result was first conjectured by Rousseau, see [6, 4.8] and [5, 2.2]).

Theorem 1.1. *Let X be a Euclidean building without factors isometric to Euclidean spaces. Let $g \in Isom(X)$ be an isometry of X. Then*

$$d_g(x) \geq \sqrt{\delta_g^2 + C \cdot d(x, \mathrm{Min}_g)^2}$$

for a constant $C > 0$ depending only on the spherical Coxeter complex associated to X and, if g is hyperbolic, on the type of the endpoint $c(\infty)$ of an axis c of g.

Notice that the conclusion of Theorem 1.1 can only be satisfied by semisimple isometries (see Proposition 2.1), that is, Euclidean buildings do not admit parabolic isometries.

2. Preliminaries

In this paper we consider Euclidean buildings from the CAT(0) viewpoint as presented in [3, Section 4], we refer to it for the basic definitions and facts about Coxeter complexes, Euclidean and spherical buildings. For more information on CAT(0) spaces in general we refer to [1].

All geodesic segments, lines and rays will be assumed to be parametrized by arc-length.

Let (X, d) be a Euclidean building. Its Tits boundary $\partial_T X$ with the *Tits distance* $\angle_T$ is a CAT(1) space admitting a unique structure (possibly trivial if X is a Euclidean space) as a *thick* spherical building modeled on a spherical Coxeter complex (S, W). We say that (S, W) is the spherical Coxeter complex associated to X. The *space of directions* or *link* $\Sigma_x X$ at a point $x \in X$