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The classification of the finite groups whose supersolvable (nilpotent) subgroups of equal order are conjugate

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Abstract

In this article the structure of any of the groups featuring in the title will be determined. As such, work of Bensaid, Lindenbergh, Sezer and van der Waall has been expanded.

Keywords: Supersolvable groups; Conjugate subgroups; Nilpotent subgroups; Sylow subgroups; Abelian subgroups

1. Introduction

All groups in this article will be finite. Notations and conventions are standard like in [3]. In the sequel, the symbol p stands for an unspecified odd prime number.

Let us consider the classes $\mathcal{B}, \mathcal{H}, \mathcal{N}$ and $\mathcal{A}$, consisting of certain groups G, as follows.

(1) $\mathcal{B} = \{G \mid$ each pair of subgroups of G of equal order consists of conjugate subgroups of $G\}$;
(2) $\mathcal{H} = \{G \mid$ each pair of supersolvable subgroups of G of equal order consists of conjugate subgroups of $G\}$;
(3) $\mathcal{N} = \{G \mid$ each pair of nilpotent subgroups of G of equal order consists of conjugate subgroups of $G\}$;

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(4) $\mathcal{A} = \{G \mid$ each pair of abelian subgroups of G of equal order consists of conjugate subgroups of $G\}$.

It is well known that abelian groups are nilpotent and that nilpotent groups are supersolvable. Thus each member of $\mathcal{B}$ belongs to $\mathcal{H}$, each member of $\mathcal{H}$ belongs to $\mathcal{N}$ and each member of $\mathcal{N}$ belongs to $\mathcal{A}$. In this article the structure of each of the groups of the classes $\mathcal{H}$ and $\mathcal{N}$ will be determined. The structure of any of the groups of the class $\mathcal{B}$ has been elucidated in a series of articles, namely [1,4,6,7] to which we kindly refer; as such, the classification is huge and full of details. The structure of any of the groups of the class $\mathcal{A}$ has been obtained in [5]. In Section 2, we recall the contents of the Main Theorem of [5] as it is needed for determining the explicit structure of anyone of the groups of the classes $\mathcal{H}$ and $\mathcal{N}$.

As to related classification problems, the reader is invited to take a look in the Introduction to [5].

2. The classification

The structure of each of the groups of the classes $\mathcal{H}$ and $\mathcal{N}$ can be revealed by means of the Main Theorem of [5]. Namely, it has been shown by Sezer and van der Waall that each member of $\mathcal{A}$ consists of a direct product of a group M and a group R of relatively prime order (i.e. $(|M|, |R|) = 1$) where M belongs to $\mathcal{B}$ and where R is isomorphic to some group from S defined by

$$S = \{\{1\};\ J_1;\ \mathrm{PSL}(2, p^f) \text{ with } f \mid 3 \text{ satisfying } p^f \geq 11 \text{ and } p^f \equiv \delta \pmod 8$$
$$\text{with } \delta \in \{-3, +3\};\ \mathrm{SL}(2, p^u) \text{ with } u \mid 3 \text{ and } p^u \geq 7\}.$$

Conversely, any choice for M from $\mathcal{B}$ and any choice for R from S lead to the fact that the direct product $M \times R$ is a member from $\mathcal{A}$ as soon as $(|M|, |R|) = 1$ is fulfilled. In particular, any member from S belongs to $\mathcal{A}$.

The next step to take, is the classification of the groups of the class $\mathcal{N}$.

The first Janko simple group J_1 of order 175 560 $(=2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19)$ has all its Sylow subgroups abelian [2, p. 150]. Hence it holds that each nilpotent subgroup of J_1 is abelian ([3], I. Aufgabe 26 together with III.2.3 Hauptsatz). Thus, as J_1 were shown to be a member from $\mathcal{A}$ in ([5], Rubric c in the proof of Theorem 4.5), we get that J_1 is a member of $\mathcal{N}$. Next, let us consider a group V occurring in S with V isomorphic to $\mathrm{PSL}(2, p^f)$. Each such V has all its Sylow subgroups abelian ([3], II.8.10 Satz). Hence V is a member of $\mathcal{N}$. Finally we focus our attention on a group U from S, isomorphic to $\mathrm{SL}(2, p^u)$. It holds that U does not belong to $\mathcal{N}$ unless 16 does not divide the order of U. [Indeed, if 16 would divide the order of U, it would imply that each Sylow 2-subgroup of U contains non-isomorphic nilpotent subgroups of order 8; consult ([3], II.8.10 Satz together with I.14.9 Satz).] Therefore we proceed with investigating the structure of those groups U from S for which 16 does not divide the order of U. It follows that $|U|$ is divisible by 8, as it is not possible that the order of U is not divisible by 4. Hence, as $|U| = p^u(p^u - 1)(p^u + 1)$, it follows that $p^u \equiv \delta \pmod 8$ with $\delta \in \{-3, +3\}$ has to hold. Any nilpotent subgroup $T \leq U$ with 8 not dividing the order of T has all its Sylow subgroups abelian ([3], I.7.5 2. Satz von Sylow, together with III.2.3. Hauptsatz). Therefore, as U has been given to be a member from $\mathcal{A}$, any $C \leq U$ and $D \leq U$ of equal order satisfying $8 \nmid |C|$, are conjugate to each other in U. There remains to examine the conjugacy property for nilpotent subgroups N_1 and N_2 of U, being of equal order for which 8 divides the order of N_1; remember also that 16 does not divide the order of N_1, due to Lagrange's Theorem applied to