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indagationes mathematicae
Indagationes Mathematicae 25 (2014) 170–185
www.elsevier.com/locate/indag

The complexification of a lattice seminormed space

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Abstract

We introduce tensor products in the category of lattice seminormed spaces. We show that the reasonable cross vector seminorms on the complexification of a lattice seminormed space are the same as the admissible vector seminorms. We then specialize these results to complexifications of Archimedean Riesz spaces.

Keywords: Lattice seminormed space; Reasonable cross norm; Tensor product; Riesz space; Complexification

1. Introduction

The primary goal of this paper is to study the complexification $E + iE$ of a Riesz space E by using tensor products in the category of lattice seminormed spaces. Extending the absolute value of a Riesz space E to a modulus on $E + iE$, if possible, opens the possibility to do complex analysis on the complexification. For a construction of such a modulus two approaches can be found in the literature. The first construction was started by de Schipper in [2] and its end product for uniformly complete Riesz spaces can be found in [10]. The essence of this construction is the formula

$$|z| = \sup\{x \cos\theta + y \sin\theta \mid \theta \in [0, 2\pi)\}$$

for $z = x + iy \in E + iE$. A second, less known, approach can be found in the paper by Mittelmeyer and Wolff [6]. There the authors provide axioms for a complex Riesz space with what they call an Archimedean modulus and then show that such a Riesz space derives from complexifying a Riesz space E and that the Archimedean modulus can be calculated via the

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0019-3577/$ - see front matter
http://dx.doi.org/10.1016/j.indag.2013.05.001

above formula. In this paper we offer a third avenue to complex Riesz spaces via tensor products. Our idea is in part motivated by van Neerven's paper [9], where the complexification of a Banach lattice is provided in terms of a tensor product. The absolute value on a Riesz space E is, in fact, a vector norm $E \to E$. We prove that there is at most one reasonable cross vector seminorm $E \otimes \mathbb{C} \to E$ when E is Archimedean. In the process, we define the projective and injective vector seminorms and show that they coincide on the complexification of a uniformly complete Riesz space. The injective vector seminorm is then linked to the de Schipper formula above, whereas other concrete admissible vector seminorms lead to other formulas. For instance, the projective vector norm leads to

$$|z| = \inf \left\{ \sum_{k=1}^{n} |\lambda_k| \, |x_k| \mid \lambda_k \in \mathbb{C}, x_k \in E, x + iy = \sum_{k=1}^{n} \lambda_k \otimes x_k \right\}.$$

The theory of lattice normed spaces has its roots in the very dawn of Riesz space theory and is very well developed in the Russian literature. However, the existence of tensor product norms on lattice normed spaces requires strong conditions on the Riesz spaces involved (see [8]). To avoid such restriction, we focus on vector *semi*norms rather than norms in this paper.

The organization of our paper is as follows. We first introduce the terminology of lattice seminormed spaces. These lattice seminormed spaces together with a notion of morphisms form a category. In that category we introduce reasonable cross vector seminorms as well as the projective and injective seminorms. Next we define the complexification of a lattice seminormed space and define admissible vector seminorms, showing them to be the same as the reasonable cross vector seminorms. In the final section of the paper we specialize the results to complexifications of Archimedean Riesz spaces. For the theory of Riesz spaces as needed in this paper we refer to [1,10], while for all things categorical we use [5].

2. Preliminary definitions

Let X be a vector space over $\mathbb{R}$ and let E be an Archimedean Riesz space. We call $p : X \to E$ a *vector seminorm* if it satisfies the following properties.

$$p(X) \subset E^+, \tag{S1}$$

$$\forall \alpha \in \mathbb{R}\ \forall x \in X \quad [p(\alpha x) = |\alpha| p(x)], \tag{S2}$$

$$\forall x, y \in X \quad [p(x + y) \le p(x) + p(y)]. \tag{S3}$$

A *lattice seminormed space* is a triple (X, E, p), written E_p^X, where X is a real vector space, E is an Archimedean Riesz space, and $p : X \to E$ is a vector seminorm.

If in addition p satisfies the property that $p^{-1}(\{0\}) = \{0\}$ then p is called a *vector norm* and E_p^X is called a *lattice normed space* (see [4]).

Seminormed vector spaces are examples of lattice seminormed spaces. Moreover, any Archimedean Riesz space E when equipped with its modulus is another example of a lattice seminormed space.

We next define dominating maps and morphisms.

Definition 2.1. Let E_p^X and F_q^Y be lattice seminormed spaces. For $T : X \to Y$ a linear map and $\bar{T} : E \to F$ a positive map, we say that $\bar{T}$ dominates T (with respect to p over q) if

$$qT \le \bar{T} p.$$