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# Jensen's and martingale inequalities in Riesz spaces

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Dedicated to the memory of A.C. Zaanen

**Abstract**

A functional calculus is defined and used to prove Jensen's inequality for conditional expectations acting on Riesz spaces. Upcrossing inequalities, martingale inequalities and Doob's $L^p$-inequality for continuous time martingales and submartingales are proved.


*Keywords:* Vector lattice; Stochastic process; Jensen's inequality; Doob's $L^p$-inequality; Doob's maximal inequality; Upcrossing theorem; Downcrossing theorem

## 1. Introduction

A.C. Zaanen played an important part in establishing the role of positivity in modern analysis. Through his books and papers (many with W.A.J. Luxemburg) he made many original contributions to the theory of positivity. With Luxemburg he standardized notions and exposed relations that existed between results obtained by workers in the different (and often isolated) schools in Russia, Japan and the USA and thus established the theory of vector lattices as we know it today. His interest in the abstract theory of positivity was firmly rooted in his earlier investigations into the theory of integral equations in Orlicz spaces. He was therefore always interested in the applications that the abstract theory had. He lived to witness many of these: applications to spectral theory of positive operators, Korotkov theory, Ergodic theory, Semi-groups of operators, to name a few. Much of this can be found in the following references: W.A.J. Luxemburg and A.C. Zaanen [19], H.H. Schaefer [22], D.H. Fremlin, [7], C.D. Aliprantis and

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O. Burkinshaw [1,2], P. Meyer-Nieberg [20] and A.C. Zaanen [26,27]. For background on Riesz spaces, we refer the reader to these texts.

Recently, the role of positivity in probability theory and stochastic processes was explored in several papers by C.C.A. Labuschagne, B.A. Watson and Wen-Chi Kuo [12–17,25], Wen-Chi Kuo [11], V. Troitsky [23] and the author in [8,9]. These ideas on stochastic processes in vector lattices were also applied with great success by S.F. Cullender and C.C.A. Labuschagne [4,5] to obtain new information about martingale convergence in Bochner spaces. These developments are not surprising, since stochastic variables are elements of functions spaces and a natural correspondence exists between stochastic variables and elements of a general vector lattice, events (i.e., sets) and order projections in the vector lattice, $\sigma$-algebras of sets and Boolean algebras of order projections. The role of a probability measure is played by a conditional expectation operator, i.e., a positive order continuous projection mapping the vector lattice into itself onto a Dedekind complete Riesz subspace (see [8,9]). Thus the theory of probability and stochastic processes relies on many abstract notions involving positivity. The present paper's contribution consists of a proof of Jensen's inequality for conditional expectations, Halmos' Optional skipping theorem, a new approach to the upcrossing theorem and the upcrossing inequality, Martingale inequalities and finally Doob's inequality. Our aim is to present these results for continuous stochastic processes and we rely on ideas presented in Karatzas and Shreve in [10] and Revuz and Yor in [21].

## 2. Notation and definitions

We assume the reader to be familiar with the basic definitions of the general theory of Riesz spaces (vector lattices) and we refer the reader to the literature cited above. For notions from Probability theory and Stochastic processes, we refer to [10,18,3,21]. We also refer the reader to the papers [8,9] by the author on continuous time stochastic processes in vector lattices, but for the convenience of the reader we repeat the main definitions.

Let $\mathfrak{E}$ be a Riesz space. The strictly positive order continuous projection $\mathbb{F} : \mathfrak{E} \to \mathfrak{E}$ is called a *conditional expectation* if

(a) $\mathfrak{F} := \mathcal{R}(\mathbb{F})$ is a Dedekind complete Riesz subspace of $\mathfrak{E}$;
(b) $\mathfrak{F}$ is weakly dense in $\mathfrak{E}$, i.e., the band generated by $\mathfrak{F}$ equals $\mathfrak{E}$.

If $\mathbb{P}$ is a projection in $\mathfrak{E}$ that maps $\mathfrak{F}$ into itself, then $\mathbb{P}$ commutes with $\mathbb{F}$. Hence, for every orthomorphism $\mathbb{S} \in \mathrm{Orth}(\mathfrak{F})$ we have that $\mathbb{S}$ commutes with $\mathbb{F}$. We shall denote the set of all order projections of $\mathfrak{E}$ that map $\mathfrak{F}$ into itself by $\mathfrak{P}_{\mathfrak{F}}$ and this set can be identified with the set of all order projections of the vector lattice $\mathfrak{F}$ (we say these projections act on $\mathfrak{F}$).

Let $T = [0, \infty)$. The family $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in T}$ is called a filtration on $\mathfrak{E}$ if $(\mathbb{F}_t)$ is a family of conditional expectations on $\mathfrak{E}$ satisfying $\mathbb{F}_s\mathbb{F}_t = \mathbb{F}_t\mathbb{F}_s = \mathbb{F}_s$ for all $s \le t$ and $\mathfrak{F}_t := \mathcal{R}(\mathbb{F}_t)$, the range of $\mathbb{F}_t$. We denote the set of all order projections in $\mathfrak{E}$ that act on $\mathfrak{F}_t$ by $\mathfrak{P}_t$ and we recall that $\mathbb{F}_t\mathbb{P} = \mathbb{P}\mathbb{F}_t$ for all $\mathbb{P} \in \mathfrak{P}_t$. For obvious reasons the projections in $\mathfrak{P}_t$ are called the *events* in the process up to time $t$.$\mathfrak{P}_t$ is a complete Boolean algebra and can be considered to play the rôle of the $\sigma$-algebra in the classical case.

A family $X = (X_t)_{t\in T}$ with $X_t \in \mathfrak{E}$ is called a (continuous time) *stochastic process* in $\mathfrak{E}$. The stochastic process $(X_t)_{t\in T}$ is *right* (resp. *left*) continuous, if o-$\lim_{s\downarrow t} X_s = X_t$ (resp.o-$\lim_{s\uparrow t} X_s = X_t$).

A stochastic process $(X_t)_{t\in T}$ is *adapted to the filtration* $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in T}$ if $X_t \in \mathfrak{F}_t$ for all $t \in T$. We write $(X_t, \mathfrak{F}_t)_{t\in T}$ to indicate that $(X_t)$ is adapted to $(\mathbb{F}_t, \mathfrak{F}_t)_{t\in T}$. The stochastic