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Measure and integration: The basic extension and representation theorems in terms of new inner and outer envelopes

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Dedicated to the memory of ADRIAAN CORNELIS ZAANEN on the occasion of the 100th anniversary of his birth

Abstract

The work of the author in measure and integration is based on new inner and outer envelope formations, which replace the traditional Carathéodory outer measure and certain simple suprema and infima. The new formations lead to essential improvements in both the extent and the adequacy of the basic results. However, they did not find an entrance into the recent textbook literature. The present paper seeks to demonstrate their power with the examples of the basic inner and outer extension and representation theorems for set functions and functionals.

Keywords: Inner and outer premeasures; Inner and outer envelopes of set functions and functionals; Inner and outer extension and representation theorems; Choquet integral; Carathéodory class

The present paper returns to the foundations of the theory developed in the author's book [1] and in the subsequent 25 articles which recently have been collected in the volume [2]. We consider the basic inner and outer $\bullet$ extension theorems for set functions, as before in the versions $\bullet = \star\sigma\tau$ with $\star =$ finite, $\sigma =$ sequential, $\tau =$ nonsequential, and the basic inner and outer $\bullet$ representation theorems for functionals in terms of the Choquet integral, this time as before for $\bullet = \sigma\tau$. The decisive formations are the new inner and outer $\bullet$ *envelopes*, which for set functions and $\bullet = \sigma\tau$ have to take the place of the Carathéodory outer measure and of the $\bullet = \star$ envelopes in the traditional treatments. The relevant results in [1,2] were drastic improvements,

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to an extent that it appears mysterious why the new concepts were not widely adopted. In the present paper the first two sections are kind of summaries of the extension theories and of the initial part of the representation theories, while the third section is a further development of the final representation theories — all with the intention to illuminate the role of our inner and outer $\bullet$ envelopes. In contrast to the earlier papers the front versions will be the *inner* ones, because there are some indications that the inner situation is the superior one. For the comparison with the traditional treatments and for concrete situations we can refer to an abundance of places in [1,2].

1. The inner and outer extension theorems

We start to recall the relevant traditional concepts. Our main references will be the [2] articles (10) and (24). The entire paper assumes a nonvoid set X, which carries the set systems under consideration. A nonvoid set system $\mathfrak{S}$ in X is called a *paving*. The most frequent ones are the *lattices* (with respect to $\cap\cup$) and the *rings* and *algebras*. For a paving $\mathfrak{S}$ we define $\mathfrak{S}_\star \subset \mathfrak{S}_\sigma \subset \mathfrak{S}_\tau$ and $\mathfrak{S}^\star \subset \mathfrak{S}^\sigma \subset \mathfrak{S}^\tau$ to consist of the intersections and unions of the finite/countable/arbitrary subpavings of $\mathfrak{S}$. One of the most fundamental concepts is that of the $\bullet$ *compact* pavings for $\bullet = \sigma\tau$, of which the counterpart $\bullet = \star$ is trivially fulfilled.

In the present context the usual set functions on a paving $\mathfrak{S}$ are the *isotone* $\varphi : \mathfrak{S} \to [0,\infty[$ or $[0,\infty]$. For these ones we recall the notions (*almost*) *downward /upward* $\bullet$ *continuous* for $\bullet = \star\sigma\tau$, of which $\bullet = \star$ is trivially fulfilled, and *inner/outer regular* $\mathfrak{M}$ for a subpaving $\mathfrak{M} \subset \mathfrak{S}$. For a set function $\varphi : \mathfrak{S} \to [0,\infty]$ on a lattice $\mathfrak{S}$ we recall the notions *modular* and *super/submodular*. In case $\varnothing \in \mathfrak{S}$ the function φ is called a *content* iff it is isotone with $\varphi(\varnothing) = 0$ and modular; this is the usual notion when $\mathfrak{S}$ is a ring.

We conclude the list of traditional concepts with the *Carathéodory class* for a set function $\vartheta : \mathfrak{P}(X) \to [0,\infty]$ with $\vartheta(\varnothing) = 0$, defined to be

$$\mathfrak{C}(\vartheta) := \{A \subset X : \vartheta(M) = \vartheta(M \cap A) + \vartheta(M \cap A') \text{ for all } M \subset X\} \subset \mathfrak{P}(X);$$

its members are called *measurable* ϑ. The basic properties of $\mathfrak{C}(\vartheta)$ are collected in [2] (24) sect.2. Beyond $\vartheta(\varnothing) = 0$ the class $\mathfrak{C}(\vartheta)$ can be defined after [1] pp. 40–42, but the explicit definition will not be needed in the sequel.

Next we recall from [2] (24) 3.1 the basic concepts for set functions in [1,2]. Let $\mathfrak{S}$ be a lattice with $\varnothing \in \mathfrak{S}$, and $\bullet = \star\sigma\tau$. We define an isotone $\varphi : \mathfrak{S} \to [0,\infty[$ with $\varphi(\varnothing) = 0$ to be an *inner* $\bullet$ *premeasure* iff it can be extended to a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}_\bullet$ such that

$\alpha|\mathfrak{S}_\bullet$ is downward $\bullet$ continuous (note that $\alpha|\mathfrak{S}_\bullet < \infty$) and

α is inner regular $\mathfrak{S}_\bullet$.

These contents α are called the *inner* $\bullet$ *extensions* of φ. Note that an inner $\bullet$ premeasure is downward $\bullet$ continuous and modular. Likewise we define an isotone $\varphi : \mathfrak{S} \to [0,\infty]$ with $\varphi(\varnothing) = 0$ to be an *outer* $\bullet$ *premeasure* iff it can be extended to a content $\alpha : \mathfrak{A} \to [0,\infty]$ on a ring $\mathfrak{A} \supset \mathfrak{S}^\bullet$ such that

$\alpha|\mathfrak{S}^\bullet$ is upward $\bullet$ continuous and

α is outer regular $\mathfrak{S}^\bullet$.

These contents α are called the *outer* $\bullet$ *extensions* of φ. Note that an outer $\bullet$ premeasure is upward $\bullet$ continuous and modular. The deviation relative to the value ∞ is of course in order to avoid the difficulties known from the traditional treatment for $\bullet = \sigma$.

These definitions produce the natural tasks to *characterize* the inner and outer $\bullet$ premeasures and to *describe* their collections of inner/outer $\bullet$ extensions. The fundamental idea to solve these