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Operator ideal properties of the integration map of a vector measure

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Abstract

Let m be a vector measure with values in a Banach space X. We explore the relationship between operator ideal properties of the integration map $I_m : f \mapsto \int f \, dm$ from $L^1(m)$ to X, and properties of the vector measure m. For instance, membership of I_m in certain operator ideals implies that $L^1(m) = L^1(|m|)$, where $|m|$ is the variation measure of m. This happens, for example, if I_m is compact, or p-summing or in certain cases, completely continuous. Characterizations of when I_m is compact or absolutely summing are also given. For many operator ideals, membership of I_m is determined solely by the range of m.

Keywords: Operator ideal; Vector measure; Range; Integration map; Finite variation

1. Introduction

The notion "Banach function space", briefly B.f.s., is synonymous with the phrase "Luxemburg and Zaanen" who were co-founders of this (by now) classical topic and who themselves made fundamental contributions over many decades. Another favourite topic of Prof. A.C. Zaanen in which he also made significant advances was the theory of integration. Given that a large and important class of B.f.s.' are the L^1-spaces of Banach-space-valued vector measures together with their associated theory of integration, it is only fitting that in a special issue dedicated to Prof. A.C. Zaanen at least one article should be on this topic.

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In this paper we explore the relationship between operator ideal properties of the integration map I_m and properties of the underlying vector measure m. In part we survey known results and, for some of them, we sketch the main ideas of their proofs. The aim is to make the paper self-contained and to use the ideas and techniques of the sketched proofs as a motivation for the new results and examples, which constitute the bulk of the paper.

Let X be a Banach space (with norm $\|\cdot\|_X$, closed unit ball $\mathbb{B}[X]$ and dual space X^*) and $m\colon \Sigma \to X$ be a *vector measure*, i.e., m is σ-additive on a σ-algebra Σ (of subsets of some set $\Omega \neq \emptyset$). A Σ-measurable function $f\colon \Omega \to \mathbb{C}$ is called *m-integrable* if

(I-1) $\int_\Omega |f|\, d|\langle m, x^*\rangle| < \infty, \forall\, x^* \in X^*$, and
(I-2) for each $A \in \Sigma$ there is $\int_A f\, dm \in X$ satisfying

$$\left\langle \int_A f\, dm, x^* \right\rangle = \int_A f\, d\langle m, x^*\rangle, \quad \forall\, x^* \in X^*;$$

see [13,14]. Here, for each $x^* \in X^*$, the scalar measure $A \mapsto \langle m(A), x^*\rangle$, for $A \in \Sigma$, is denoted by $\langle m, x^*\rangle$. The space of m-integrable functions is denoted by $L^1(m)$; it is identified with its quotient space modulo m-null functions, where $f \in L^1(m)$ is *m-null* if $\int_A f\ dm = 0$, for all $A \in \Sigma$. In fact, f is m-null if and only if $|m|(\{f \neq 0\}) = 0$, where $|m| : \Sigma \to [0, \infty]$ is *the variation measure* of m, defined analogously as for scalar measures, [6, Definition I.1.4]. The space $L^1(m)$ is complete for the *lattice norm*

$$\|f\|_{L^1(m)} := \sup_{x^* \in \mathbb{B}[X^*]} \int_\Omega |f|\, d|\langle m, x^*\rangle|, \quad \forall\, f \in L^1(m), \tag{1.1}$$

i.e., $L^1(m)$ is a (complex) Banach lattice, simple functions are dense in $L^1(m)$, and the *integration map $I_m\colon L^1(m) \to X$*, defined by

$$I_m\colon f \mapsto \int_\Omega f\, dm, \quad \forall\, f \in L^1(m), \tag{1.2}$$

is linear, continuous and has operator norm $\|I_m\|_{\mathrm{op}} = 1$, [22, p. 152]. The norm (1.1) is order continuous and, relative to any control measure for m, the space $L^1(m)$ is a B.f.s., [22, Ch. 3]. Bounded measurable functions are m-integrable. In fact, for the Banach space $L^\infty(m) = L^\infty(|m|)$ of all (equivalence classes of) bounded Σ-measurable functions, equipped with the essential sup-norm $\|\cdot\|_{L^\infty(m)}$, we have the continuous inclusion $L^\infty(m) \subseteq L^1(m)$, [14, p. 161]. The restriction of I_m to $L^\infty(m)$ is called the *Bartle integral*, [6, Definition I.1.12]. We denote this restriction by $I_m^{(\infty)}$. The Bartle integral is useful for representing weakly compact operators defined on $C(K)$ spaces, [6, Ch. VI]. The following result summarizes certain ideal properties of $I_m^{(\infty)}$. Most of the proof, perhaps in a slightly different setting, occurs in [6, Chapters I, II & VI]. For the notion of an X-valued Bochner μ-integrable function on Ω, relative to a finite measure $\mu \geq 0$ on Σ, we refer to [6, Ch. II, Section 2]; see also Section 3 below.

Theorem 1.1. *Let X be a Banach space and $m : \Sigma \to X$ be a vector measure.*

(1) *The Bartle integral $I_m^{(\infty)} : L^\infty(m) \to X$ is a weakly compact operator.*
(2) *$I_m^{(\infty)}$ is compact if and only if the range of m is relatively compact in X.*
(3) *$I_m^{(\infty)}$ is absolutely summing if and only if m has finite variation.*
(4) *$I_m^{(\infty)}$ is nuclear if and only if m has finite variation and admits a Bochner derivative $\frac{dm}{d|m|}$: $\Omega \to X$ relative to $|m|$.*