Available online at www.sciencedirect.com
ScienceDirect
Indagationes Mathematicae 25 (2014) 366–375
indagationes mathematicae
www.elsevier.com/locate/indag

Cone isomorphisms and almost surjective operators

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Abstract

Let E be a Banach lattice and F a Banach space. A bounded linear operator $T : E \to F$ is an isomorphism on the positive cone of E if and only if T^* is almost surjective. A dual version of this theorem holds also. A bounded linear operator $T : F \to E$ is almost surjective if and only if T^* is an isomorphism on the positive cone of F^*.

Keywords: Cone isomorphism; Banach lattice

0. Introduction

It is well known that if E and F are Banach spaces and $T : E \to F$ is a bounded linear operator, then T is an isomorphism onto a subspace of F if and only if the adjoint linear operator T^* is surjective and dually T is surjective if and only if T^* is an isomorphism onto a subspace of E^* (see e.g. Theorem 3.1.22 of [12]). In this paper we will study in Section 1 the analogues to this result by taking one of the two Banach spaces to be a Banach lattice and by requiring that the corresponding operator defined on the Banach lattice is only an isomorphism on the positive cone of the Banach lattice. It turns out that we can replace the surjectivity condition by a weaker condition which we call *almost surjectivity*. We call a bounded linear operator T from a Banach space F into a Banach lattice E almost surjective, if for all $0 \le y \in E$ there exists $x \in F$ such that $y \le Tx$. In case E is also a Banach lattice and T is a positive linear map, then this condition is equivalent to the order ideal generated by the range of T being equal

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doi:10.1016/j.indag.2013.01.004

to E. Our study of cone isomorphisms is motivated by notions introduced in the context of vector measures. We briefly recall the connection. For more information and examples we refer to [3,4,6,13,14]. Let F be a Banach lattice and $\nu : X \to F$ be a vector measure generated by a certain kernel operator T with values in F, where $\nu(A) = T(\chi_A)$. One can define an optimal domain $[T, F]$ such that $T : [T, F] \to F$, with $[T, F]$ a Banach function space and the norm on $[T, F]$ is given by $\|T(|f|)\|_F$. Hence, such operators are by definition cone isometries (cf. Definition 1.4) on their optimal domain. It is in this context that we discovered in [15] a special case of Theorem 1.6 below. Here we will prove that a bounded linear operator T from a Banach lattice into a Banach space is a cone isomorphism if and only the adjoint operator T^* is almost surjective. Then we will establish the dual result that a bounded linear operator T from a Banach space into a Banach lattice is almost surjective if and only if the adjoint operator T^* is a cone isomorphism. In Section 2 we then consider positive linear operators from L_p-spaces into Banach lattice and discuss the relation between positive cone isomorphisms (isometries) and positive isomorphisms. We prove e.g. that if $0 \le T : L_p \to E$ is cone isometry, with $1 < p < \infty$, and E having strictly monotone norm, then T is a lattice isometry. Finally in the last section we apply some of the obtained results to optimal domains of positive operators.

1. Cone isomorphisms

Let E be a Banach lattice and F be a Banach space.

Definition 1.1. A bounded linear operator $T : F \to E$ is called *almost surjective* if for all $0 \le y \in E$ there exists $x \in F$ such that $y \le Tx$.

Note that if E and F are Banach lattices and T is a positive linear operator then T is almost surjective if and only if for all $y \in E$ there exists $x \in F$ such that $|y| \le |Tx| \le T|x|$. Hence, a positive T is almost surjective if and only if the ideal generated by the range of T equals E. To investigate almost surjective operators we introduce a notation. Let $T : F \to E$ be a bounded linear operator. Then $E_T = \{y \in E : |y| \le Tx \text{ for some } x \in F\}$. In particular, T is almost surjective if and only if $E_T = E$. Next we introduce a norm on E_T by defining $\|y\|_T = \inf\{\|x\|_F : |y| \le Tx\}$ for $y \in E_T$. Note that $\|y\|_E \le \|T\|\|y\|_T$ for all $y \in E_T$.

Proposition 1.2. *Let E be a Banach lattice, F be a Banach space and let $T : F \to E$ be a bounded linear operator. Then $\|\cdot\|_T$ is a lattice norm on E_T and E_T is complete with respect to this norm.*

Proof. We first show that $\|\cdot\|_T$ is a norm on E_T. It is clear that $\|y\|_T \ge 0$ for all $y \in E_T$. Assume that $\|y\|_T = 0$. Then $\|y\|_E \le \|T\|\|y\|_T$ on E_T implies that $\|y\|_E = 0$, so $y = 0$. It is straightforward to verify for $\|y\|_T$ the other properties of a norm and it is obviously a lattice norm. It remains to show that E_T is complete with respect to the norm $\|\cdot\|_T$. Let $y_n \in E_T$ satisfy $\sum_{n=1}^{\infty} \|y_n\|_T < \infty$. Then $\sum_{n=1}^{\infty} \|y_n\|_E < \infty$, so there exists $0 \le y_0 \in E$ such that $y_0 = \sum_{n=1}^{\infty} |y_n|$, where the series converges in the norm of E. Now for every $n \ge 1$ we can find $x_n \in F$ such that $|y_n| \le Tx_n$ and $\|x_n\| \le \|y_n\|_T + \frac{1}{2^n}$. From $\sum_{n=1}^{\infty} \|x_n\| < \infty$, it follows that there exists $x_0 \in F$ such that $x_0 = \sum_{n=1}^{\infty} x_n$, where the series converges in norm. Now $Tx_0 = \sum_{n=1}^{\infty} Tx_n \ge \sum_{n=1}^{M} |y_n|$ for all $M \ge 1$, implies that $Tx_0 \ge y_0$. Hence, $y_0 \in E_T$. It remains to show that the series defining y_0 converges with respect to the norm of E_T. Let $N \ge 1$.