Available online at www.sciencedirect.com

ScienceDirect

Indagationes Mathematicae 25 (2014) 376–388

indagationes mathematicae

www.elsevier.com/locate/indag

Completeness of quasi-normed symmetric operator spaces☆

Fedor Sukochev

School of Mathematics and Statistics, University of New South Wales, Sydney, 2052, Australia

Abstract

We show that (generalized) Calkin correspondence between quasi-normed symmetric sequence spaces and symmetrically quasi-normed ideals of compact operators on an infinite-dimensional Hilbert space preserves completeness. We also establish a semifinite version of this result.

Keywords: Symmetric operator spaces; Quasi-Banach spaces

1. Introduction and preliminaries

Let H be a complex Hilbert space, $B(H)$ be the algebra of all bounded linear operators on H and let $\mathcal{C}_0(H)$ be the ideal of all compact operators. There is a remarkable correspondence between sequence spaces and two-sided ideals of compact operators due to Calkin [1]. Recall, that a linear subspace $\mathcal{J}$ of compact operators is a two-sided ideal if $X \in \mathcal{J}$ and $Y \in B(H)$ implies $YX, XY \in \mathcal{J}$. A Calkin space J is a subspace of c_0 (the space of all vanishing sequences) such that $x \in J$ and $\mu(y) \leq \mu(x)$ implies $y \in J$, where $\mu(x)$ is the decreasing rearrangement of the sequence $|x|$. For a compact operator X, an eigenvalue sequence

$$\lambda(X) := \{\lambda(n, X)\}_{n=1}^{\infty} \in c_0$$

is the sequence of eigenvalues $\lambda(n, X)$, $n = 1, 2, \ldots$, of X, repeated according to algebraic multiplicity, such that the absolute values $|\lambda(n, X)|$ are decreasing. The singular value sequence

☆ Research supported by the Australian Research Council.
E-mail address: f.sukochev@unsw.edu.au.

0019-3577/$ - see front matter
doi:10.1016/j.indag.2012.05.007

$\mu(X)$ of the operator X is the eigenvalue sequence of $|X| = \sqrt{X^*X}$, that is

$$\mu(X) := \{\mu(n, X)\}_{n=1}^{\infty} = \{\lambda(n, |X|)\}_{n=1}^{\infty}.$$

The Calkin correspondence may be explained as follows. If J is a Calkin space then associate to it the subset $\mathcal{J}$ of compact operators

$$\mathcal{J} := \{X \in \mathcal{C}_0(H) : \mu(X) \in J\}.$$

Conversely, if $\mathcal{J}$ is a two-sided ideal, then associate to it the sequence space

$$J := \{x \in c_0 : \mu(x) = \mu(X) \text{ for some } X \in \mathcal{J}\}.$$

For the proof of the following theorem we refer to Calkin's original paper, [1], and to Simon's book, [20, Theorem 2.5].

Theorem 1 (*Calkin Correspondence*)**.** *The correspondence $J \leftrightarrow \mathcal{J}$ is a bijection between Calkin spaces and two-sided ideals of compact operators.*

In the recent paper [12], due to Kalton and the author, this correspondence has been specialized to symmetrically-normed ideals [7,8,20] and symmetric sequence spaces [15]. We use the notation $\|\cdot\|_\infty$ to denote the uniform norm on $B(H)$.

Definition 2. (i) An ideal $\mathcal{E}$ of $B(H)$ is said to be symmetrically (quasi)-normed if it is equipped with a Banach (quasi)-norm $\|\cdot\|_{\mathcal{E}}$ such that

$$\|XY\|_{\mathcal{E}}, \|YX\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}}\|Y\|_\infty, \quad X \in \mathcal{E}, Y \in B(H).$$

(ii) A Calkin space E is a symmetric sequence space if it is equipped with a Banach (quasi)-norm $\|\cdot\|_E$ such that $\|y\|_E \leq \|x\|_E$ for every $x \in E$ and $y \in c_0$ such that $\mu(y) \leq \mu(x)$.

For convenience of the reader, we recall that a map $\|\cdot\|$ from a linear space X into the field $\mathbb{R}$ of real numbers is a quasi-norm, if for all $x, y \in X$ and scalars α the following properties hold:

(i) $\|x\| \geqslant 0$, $\|x\| = 0 \Leftrightarrow x = 0$;
(ii) $\|\alpha x\| = |\alpha|\|x\|$;
(iii) $\|x + y\| \leqslant C(\|x\| + \|y\|)$, $C \geqslant 1$.

The couple $(X, \|\cdot\|)$ is a quasi-normed space and the least constant C satisfying the inequality (iii) above is called the modulus of concavity of the quasi-norm $\|\cdot\|$ and denoted by C_X. A complete a quasi-normed space is called quasi-Banach.

It easily follows from Definition 2 that if $(\mathcal{E}, \|\cdot\|_{\mathcal{E}})$ is a symmetrically (quasi)-normed ideal, $X \in \mathcal{E}$ and $Y \in B(H)$ is such that $\mu(Y) \leq \mu(X)$, then $Y \in \mathcal{E}$ and $\|Y\|_{\mathcal{E}} \leq \|X\|_{\mathcal{E}}$. In particular, it is easy to see that if E is Calkin space corresponding to $\mathcal{E}$, then setting $\|x\|_E := \|X\|_{\mathcal{E}}$ (where $X \in \mathcal{E}$ is such that $\mu(x) = \mu(X)$) we obtain that $(E, \|\cdot\|_E)$ is a symmetric sequence space. The converse implication is much harder [12]. To state the main result of [12], we need to recall the notion of p-convexity in symmetric spaces. Suppose $0 < p < \infty$. Then a quasi-Banach sequence space E is said to be *p-convex* where $0 < p < \infty$ if there is a constant C so that

$$\|(|x_1|^p + \cdots + |x_n|^p)^{1/p}\|_E \leq C(\|x_1\|_E^p + \cdots + \|x_n\|_E^p)^{1/p}, \qquad x_1, \ldots, x_n \in E.$$

The following theorem is a combination of [12, Theorem 8.7] and [12, Theorem 8.11] restated here for a special case of ideals of compact operators. Let $(E, \|\cdot\|_E)$ be a symmetric quasi-Banach