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Estimating Mahler measures using periodic points for the doubling map

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Abstract

The Mahler measure $m(P)$ of a polynomial P is a numerical value which is useful in number theory, dynamical systems and geometry. In this article we show how this can be written in terms of periodic points for the doubling map on the unit interval. This leads to an interesting algorithm for approximating $m(P)$ which we illustrate with several examples.

Keywords: Mahler measure; Periodic points; Numerical approximation

1. Introduction

The Mahler measure $m(P)$, or height, of a polynomial $P(z) \in \mathbb{C}[z]$ is a numerical value which plays an important role in number theory, dynamical systems and geometry [10]. It is named after the mathematician Kurt Mahler, by whom it was introduced to provide a simple proof of Gelfond's inequality for the product of polynomials in many variables. However, it has been used with great success in a variety of settings. For example, it was also used previously in Lehmer's investigation of certain cyclotomic functions and led him to ask his famous question (now known as Lehmer's conjecture) about which polynomial has the least Mahler measure. It is conjectured to be $x^{10}+x^9-x^7-x^6-x^5-x^4+x^3+x+1$ [8], but in spite of considerable work by many authors on this conjecture, it has not yet been proved. In another direction, connections have been found between the Mahler measure of certain two variable polynomials and invariants of hyperbolic 3-manifolds such as the volume, via the dilogarithm and work of Milnor, Zagier and others [11,16]. Last, but not least, the Mahler measure of an integer polynomial in k variables gives the

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topological entropy of certain $\mathbb{Z}^k$-dynamical system (for $k \geq 2$) canonically associated to the polynomial in the work of Lind, Schmidt and Ward [9]. These can be viewed as generalizations of Yuzvinskii's formula for automorphisms of solenoid's in the case $k = 1$. However, unlike the case of $\mathbb{Z}$-actions these entropies need not be the logarithms of algebraic integers.

There exist more classical approaches to estimating the Mahler measure, including relating it to L-functions, as in the work of Smyth [14]. However, in this note we want to give a new explicit formula for the Mahler measure. This is in terms of a rapidly convergent series, the terms of which are explicitly given in terms of the values of the polynomial at a finite set of points. However, perhaps the most satisfying aspect of this approach is that we use periodic points of a simple dynamical system (the doubling map on the interval) to describe the series, particularly given that one of the recent applications described above is to computing the entropy of $\mathbb{Z}^2$-dynamical systems.

Theorem 1.1. *We can write the logarithmic Mahler measure* $\log m(P)$ *as an infinite series*

$$\log m(P) = \sum_{n=1}^{\infty} a_n$$

where

1. *the terms* a_n *are explicitly given in terms of the values of the polynomial* P *at the points* $\{\frac{k}{2^n-1} : k = 0, 1, \ldots, 2^n - 1\}$*; and*
2. *there exist* $0 < \theta < 1$ *and* $C > 0$ *such that* $|a_n| \leq C\theta^{n^2}$.

The use of infinite series as explicit expressions for the Mahler measure of polynomials is not new, occurring as it does (for example) in the work of Smyth and others. In that case, it is related to expressions in terms of L-functions, i.e., generalizations of zeta functions in number theory. Our approach is based on another class of complex functions $d(z, s)$, which are closely related to dynamical zeta functions.

1.1. Overview of the proof of Theorem 1.1

One can write $\log m(P)$ in terms of the integral of $\log |P(\cdot)|$ around the unit circle. In the special case that the polynomial $P(z)$ is non-zero on the unit circle then the function $\theta \mapsto \log |P(e^{i\theta})|$ is real analytic on the interval $[0, 2\pi]$. In particular, Theorem 1.1 follows by a direct application of the method in [6] (recalled in Section 4). In the general case we simply need to replace the integral in the expression for $m(P)$ by sum of two integrals, i.e., writing $P(z) = P_0(z)P_1(z)$ where $P_0(z)$ has no zeros on the unit circle and $P_1(z) = \prod_{j=1}^{m} |z - e^{2\pi i\theta_j}|$, where $e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_m}$ are the zeros of $P(z)$ on the unit circle, we can then trivially write

$$\int_0^{2\pi} \log |P(e^{2\pi i\theta})| d\theta = \int_0^{2\pi} \log |P_0(e^{2\pi i\theta})| d\theta + \int_0^{2\pi} \log |P_1(e^{2\pi i\theta})| d\theta. \tag{1.1}$$

The method in [6] now applies to estimate the first integral $\int_0^{2\pi} P_0(e^{2\pi i\theta}) d\theta$. The second integral makes no contribution, i.e.,

$$\int_0^{2\pi} \log |P_1(e^{i\theta})| d\theta = m \int_0^{2\pi} \log |1 - e^{i\theta}| d\theta = 0$$

(cf. [4, pp. 8–9]). In particular, we obtain the series expansion for $\log m(P)$ by carrying out the expansion for the first term on the right hand side of (1.1).