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# Shape of matchbox manifolds

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**Abstract**

In this work, we develop shape expansions of minimal matchbox manifolds without holonomy, in terms of branched manifolds formed from their leaves. Our approach is based on the method of coding the holonomy groups for the foliated spaces, to define leafwise regions which are transversely stable and are adapted to the foliation dynamics. Approximations are obtained by collapsing appropriately chosen neighborhoods onto these regions along a "transverse Cantor foliation". The existence of the "transverse Cantor foliation" allows us to generalize standard techniques known for Euclidean and fibered cases to arbitrary matchbox manifolds with Riemannian leaf geometry and without holonomy. The transverse Cantor foliations used here are constructed by purely intrinsic and topological means, as we do not assume that our matchbox manifolds are embedded into a smooth foliated manifold, or a smooth manifold.


*Keywords:* Solenoids; Matchbox manifold; Laminations; Delaunay triangulations

## 1. Introduction

In this work, we consider topological spaces $\mathfrak{M}$ which are continua; that is, compact, connected metric spaces. We assume that $\mathfrak{M}$ has the additional structure of a codimension-zero foliated space, so are *matchbox manifolds*. The path-connected components of $\mathfrak{M}$ form the leaves of a foliation $\mathcal{F}$ of dimension $n \geq 1$. The precise definitions are given in Section 2 below. Matchbox manifolds arise naturally in the study of exceptional minimal sets for foliations of compact manifolds, and as the tiling spaces associated to repetitive, aperiodic tilings of Euclidean space

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$\mathbb{R}^n$ which have finite local complexity. They also arise in some aspects of group representation theory and index theory for leafwise elliptic operators for foliations, as discussed in the books [15,42].

The class of Williams solenoids and the results about these spaces provide one motivation for this work. Recall that an expanding attractor $\Lambda$ for an Axiom A diffeomorphism $f: M \to M$ of a compact manifold is a continuum; that is, a compact, connected metric space. Williams developed a structure theory for these spaces in his seminal works [59–61]. The hyperbolic splitting of the tangent bundle to $TM$ along $\Lambda$ yields a foliation of the space $\Lambda$ by leaves of the expanding foliation for $f$, and the contracting foliation for $f$ gives a transverse foliation on an open neighborhood $\Lambda \subset U \subset M$. Williams used this additional structure on a neighborhood of $\Lambda$ to obtain a "presentation" of $\Lambda$ as an inverse limit of "branched $n$-manifolds", $\widehat{f}: M_0 \to M_0$, where $n$ is the dimension of the expanding bundle for $f$, and the map $f$ induces the map $\widehat{f}$ between the approximations. The notion of a 1-dimensional branched manifold is easiest to define, as the branches are required only to meet each other at disjoint vertices. In higher dimensions, the definition of branched manifolds becomes more subtle, and especially for the "transversality condition" imposed on the cell attachment maps. The spaces $\Lambda$ with this structure are called *Williams solenoids* in the literature. The topological properties of the approximating map $\widehat{f}: M_0 \to M_0$ are used to study the dynamical system defined by $f$, for example as discussed in the works [22,36,53,55] and others.

The Riemann surface laminations introduced by Sullivan [54] are compact topological spaces locally homeomorphic to a complex disk times a Cantor set, and a similar notion is used by Lyubich and Minsky in [40], and Ghys in [29]. These are also well-known examples of matchbox manifolds.

Associated to the foliation $\mathcal{F}$ of a matchbox manifold $\mathfrak{M}$ is a compactly generated pseudogroup $\mathcal{G}_{\mathcal{F}}$ acting on a "transverse" totally disconnected space $\mathfrak{X}$, which determines the transverse dynamical properties of $\mathcal{F}$. The first two authors showed in [16] that if the action of $\mathcal{G}_{\mathcal{F}}$ on $\mathfrak{X}$ is equicontinuous, then $\mathfrak{M}$ is homeomorphic to a weak solenoid (in the sense of [24,41,51]). Equivalently, the shape of $\mathfrak{M}$ is defined by a tower of proper covering maps between compact manifolds of dimension $n$.

The purpose of this work is to study the shape properties of an arbitrary matchbox manifold $\mathfrak{M}$, without the assumption that the dynamics of the associated action of $\mathcal{G}_{\mathcal{F}}$ is equicontinuous, though with the assumption that $\mathfrak{M}$ is minimal; that is, that every leaf of $\mathcal{F}$ is dense. We also assume that $\mathcal{F}$ is without holonomy. Our main result shows that all such spaces have an analogous structure as that of a William solenoid. An important difference between the case of Williams solenoids, and the general case we consider, is that the tower of approximations is not in general defined by a single map, but uses a sequence of maps between compact branched manifolds.

A *presentation* of a space $\Omega$ is a collection of continuous maps $\mathcal{P} = \{p_\ell: M_\ell \to M_{\ell-1} \mid \ell \geq 1\}$, where each $M_\ell$ is a connected compact branched $n$-manifold, and each $p_\ell: M_\ell \to M_{\ell-1}$ is a proper surjective map of branched manifolds, as defined in Section 10. It is assumed that there is given a homeomorphism $h$ between $\Omega$ and the inverse limit space defined by

$$\mathcal{S}_{\mathcal{P}} \equiv \varprojlim \{p_\ell: M_\ell \to M_{\ell-1}\}. \tag{1}$$

A *Vietoris solenoid* [57] is a 1-dimensional solenoid, where each $M_\ell$ is a circle, and each $p_\ell: \mathbb{S}^1 \to \mathbb{S}^1$ is a covering map of degree greater than 1. More generally, if each $M_\ell$ is a compact manifold and each $p_\ell$ is a proper covering map, then we say that $\mathcal{S}_{\mathcal{P}}$ is a *weak solenoid*, as discussed in [24,41,51].