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Tilings for Pisot beta numeration

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Abstract

For a (non-unit) Pisot number β, several collections of tiles are associated with β-numeration. This includes an aperiodic and a periodic one made of Rauzy fractals, a periodic one induced by the natural extension of the β-transformation and a Euclidean one made of integral beta-tiles. We show that all these collections (except possibly the periodic translation of the central tile) are tilings if one of them is a tiling or, equivalently, the weak finiteness property (W) holds. We also obtain new results on rational numbers with purely periodic β-expansions; in particular, we calculate $\gamma(\beta)$ for all quadratic β with $\beta^2 = a\beta + b$, $\gcd(a, b) = 1$.

Keywords: Beta-expansion; Pisot number; Tiling; Rauzy fractal

1. Introduction

The investigation of tilings generated by beta numeration began with the ground work of Thurston [35] who produced Euclidean tilings as geometrical picture of the expansion of numbers in a Pisot unit base β. These tilings are particular instances of substitution tilings, which were introduced by Rauzy in the seminal paper [28]. Since then, a large theory for irreducible unimodular Pisot substitutions has been developed; see e.g. the surveys [8,11]. Nevertheless, it is still an open question whether each irreducible unimodular Pisot substitutions naturally defines a

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tiling (and not a multiple covering) of the respective representation space. This question, known in this context as Pisot conjecture, is related to many different branches of mathematics, such as spectral theory, quasicrystals, discrete geometry and automata. Note that β-substitutions can be reducible and that the Pisot conjecture does not hold for reducible substitutions; see e.g. [6]. However, no example of a β-substitution failing the Pisot conjecture is known.

The aim of the present paper is to study tilings associated with beta numeration in the context of Pisot numbers that are not necessarily units. The space where these tilings are represented consists of a suitable product of Archimedean and non-Archimedean completions of the number field $\mathbb{Q}(\beta)$. The study of substitution tilings in the non-unit case started in [31], and further advances were achieved e.g. in [33,9,6,24]. In [4], the focus is given in particular on the connection between purely periodic β-expansions and Rauzy fractals.

In the present paper, we discuss several objects: *Rauzy fractals*, *natural extensions*, and *integral beta-tiles*. We recall in Theorem 1 some of the main properties of Rauzy fractals associated with beta numeration. It is well known that they induce an aperiodic multiple tiling of their representation space, and there are several topological, combinatorial, and arithmetical conditions that imply the tiling property. In the irreducible unit context, having an aperiodic tiling is equivalent to having a periodic one [22]. The situation is different when we switch to the reducible and non-unit cases. In order to have a periodic tiling, a certain algebraic hypothesis (QM), first introduced in [32] for substitutions, must hold, and our attention is naturally restricted to a certain *stripe space* when dealing with the non-unit case.

Another big role in the present paper is played by the natural extension of the β-shift. Recall that the natural extension of a (non-invertible) dynamical system is an invertible dynamical system that contains the original dynamics as a subsystem and that is minimal in a measure theoretical sense; it is unique up to metric isomorphism. If β is a Pisot number, then we obtain a geometric version of the natural extension of the β-shift by suspending the Rauzy fractals; see Theorem 2. This natural extension domain characterises purely periodic β-expansions [19,21,9] and forms (in the unit case) a Markov partition for the associated hyperbolic toral automorphism [27], provided that it tiles the representation space periodically. The Pisot conjecture for beta numeration can be stated as follows: the natural extension of the β-shift is isomorphic to an automorphism of a compact group.

In the non-unit case, a third kind of compact sets, studied in [10] in the context of shift radix systems and similar to the intersective tiles in [34], turns out to be interesting. Integral beta-tiles are Euclidean tiles that can be seen as “slices” of Rauzy fractals. In Theorem 3, we provide some of their properties. In particular, we show that the boundary of these tiles has Lebesgue measure zero; this was conjectured in [10, Conjecture 7.1].

One of the main results of this paper is the equivalence of the tiling property for all our collections of tiles. We extend the results from [22] to the beta numeration case (where the associated substitution need not be irreducible or unimodular), with the restriction that the quotient mapping condition (QM) is needed for a periodic tiling with Rauzy fractals. Our series of equivalent tiling properties also contains that for the collection of integral beta-tiles. We complete then Theorem 4 by proving the equivalence of these tiling properties with the weak finiteness property (W), and with a spectral criterion concerning the so-called boundary graph.

Finally, we make a thorough analysis of the properties of the number-theoretical function $\gamma(\beta)$ concerning the purely periodic β-expansions. This function was defined in [2] and is still not well understood; see [1], but note that the definition therein differs from ours for non-unit algebraic numbers. We improve in Theorem 5 some results of [4] and answer in Theorem 6 some of their posed questions for quadratic Pisot numbers.